THE MATHEMATICS AND COMPUTER GRAPHICS OF SPIRALS IN PLANTS

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Abstract—This is a study of the pattern known as spiral phyllotaxis—literally, “leaf-arrangement”, but applied to the arrangement of seeds, florets, petals, scales, twigs and so on—and which is very widespread in plants. A number of stages in the refinement of model-making are recorded. In particular, the model illustrated in the literature on the subject indicating uniform growth is taken a step further in generality to indicate non-uniform growth.

The intention has been to arrive at a wholly visual statement. And yet the mathematical approach is integral to this end. The study of flower forms becomes a study of the relation between the circle and the golden ratio.

A computerised drawing system was used to draw forms that are algebraically defined. This solved a technical problem, but also gave me a glimpse of the enormous scope opened up by this electronic tool. To draw, for example, a simple equi-angular spiral (as found in snail shells) is difficult and time-consuming by traditional means, but elementary by computer.

This is a study of what I call the “green spiral”, the spiral phyllotaxis—literally, “leaf arrangement”, but applied to the arrangement of seeds, florets, petals, scales, twigs, and so on—and which is very widespread in plants. I began by painting an idealised daisy based on the pattern shown in Fig. 1. It is found in Islamic art and in Escher’s artwork. The following are its principles of construction: On a circle of any radius twelve smaller and identical circles may be drawn such that they touch. They will bear an exact ratio [1] in size with their host circle. And then an exactly similar necklace of circles can be drawn to touch the first—inside the first or outside. Each circle nests between two circles of a neighbouring necklace and bears an exact ratio [2] in size with circles of a neighbouring necklace. This ratio of increased size remains uniform and the pattern may be extended forever outwards, or forever inwards. It is a pattern of growth.

This pattern of uniform growth is that of a geometric series e.g. 1, 2, 4, 8, 16, …, or 5, 15, 45, 135, … . In general $s_i = k s_{i-1}$ for some fixed $k$, giving $s_i = s_0 k^i$.

The patterns found in plants, unlike the static regular lattices of crystalline structures, must exhibit members of a sequence at different stages of growth. The pattern in Fig. 1 offers a close-packing of twelve identical sequences growing uniformly.

Spiral and helical phyllotaxis is an archetypal arrangement studied in the patterns of sequential branching around plant stems. Pine cones, daisies, sunflowers, pineapples and cacti are some of the clearest examples. Twigs, leaves, fruits or florets may constitute the elements of the sequence. One possible gestalt of Fig. 1 is the pattern of opposing spirals, twelve in each direction. Analogous spirals are seen in daisies (Fig. 2).

However, the spirals seen in plants differ in number clockwise [3]! Indeed the numbers thus exhibited are consecutive pairs taken from the celebrated Fibonacci series [4]: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … . The $n$th term of the series is the sum of the two previous terms, $f_n = f_{n-1} + f_{n-2}$; $f_0 = 0, f_1 = 1$.

The cone of a redwood presents to the eye a pattern of 3 X 5 “helico-spirals”. The daisy and sunflower offer 21 X 34 and 34 X 55.

Therefore, returning to the drawing board: about a single pole 21 X 34 spirals were drawn (programme 1) (Fig. 3). Each spiral has the form $r=ab^\theta$, a logarithmic spiral. Over equal angular increments the spiral grows by a fixed proportion. It is also called the exponential spiral or the proportional spiral or the equi-angular spiral. The last name may be explained like this: At any point on the spiral, the angle between the tangent (the momentary direction of the spiral) and the direction to the “centre” of the spiral remains the same. It is the spiral found in snail shells. The uniform rate of growth is determined by the constant $b$. Making $b$ small tightens the spiral, degenerating to a circle when $b = 1$. The spiral approaches a straight ray when $b$ becomes large.

In order for a differing number of spirals in each direction to give rise to an array of approximately regular and tessellating rhombi, each set is given a different and appropriate value for $b$.

This pattern, however, gives rise only to uniform growth. The seeds go on growing by the same ratio without limit. Clearly no sunflower does this. The seeds must slow and cease enlargement, and yet maintain packing. Are the Fibonacci spirals lost as this happens? Intriguingly, no. What seems to happen [5] is that the ratios switch over from 8 X 13 to 13 X 21, or from 21 X 34 to 34 X 55!

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Fig. 4. The golden rectangle, showing the recursive pattern of repeatedly subtracting the square which fits the shorter side to leave another, smaller, golden rectangle.

It is necessary to look at the pattern another way. Why are the ratios Fibonacci ratios in any case? This is where the golden ratio \( T \) comes in. The golden ratio is often presented in the form of a rectangle (Fig. 4). The golden rectangle is the only one so proportioned that it permits of the following algorithm: From the rectangle, cut off the square which fits the shorter side. In a golden rectangle and only in a golden rectangle this step leaves an exactly similar rectangle. Thus the step may be repeated, over and over.

The diagram, as well as illustrating recursion, gnomonic growth and nesting, also provides an aid to seeing what the numerical value of \( r \) must be. Taking the long side = \( T \), short side = 1, the first rectangle left when the square is removed will have sides measuring 1 and \( r - 1 \). Equating the proportions of the two golden rectangles: \( \frac{1}{r} = \frac{T-1}{1} \), giving \( r^2 = r + 1 \).

This has two roots, \( r = \frac{1+\sqrt{5}}{2} = 1.618034 \ldots \) and \( r' = \frac{1-\sqrt{5}}{2} = -0.618034 \ldots \)

Note \( -r' = r - 1 \) and \( -\frac{1}{r'} = r - 1 \).

The algorithm of the golden rectangle also provides a simple demonstration that \( r \) is irrational. The argument is by reductio ad absurdum. For if \( r = \frac{a}{b} \), where \( a \) and \( b \) are finite whole numbers, then a rectangle measuring \( a \times b \) is golden. The shorter side of the first remaining rectangle will be \( a - b \), a smaller finite whole number. The shorter side of the second remaining rectangle will be an even smaller whole number. And so on ad infinitum!

Expressed in terms of a line, a given line segment is cut according to the golden ratio if the larger to the smaller part bears the same ratio of size as the whole to the larger part.

\( r \) is found in the proportions of a regular pentagon. It is the ratio of lengths of a diagonal to a side!

What is less widely appreciated is the place of \( T \) in the circle. Divide the circumference of a circle into two arcs bearing the golden ratio. Two cuts are required which may be denoted 0 and 1. This is analogous to the golden section of a line segment. The ratio of larger arc to smaller is the same as the whole circle to larger arc (Fig. 5(a)). Now take this division of the circle to define the "golden" angle, between the directions of 0 and 1. This angle is \( r^2 \) whole turns, equaling 137.50776\(^\circ\). (Notice that the opposite turn from 0 to 1, \( r^{-1} = \frac{222.49224 \ldots}{5} \), amounts to the same operation.)

This angle provides the following "golden" algorithm of the circle: 0 denotes a starting direction, 1 denotes the direction after one turn of the golden angle \( (r^2) \) turns, 2 denotes the direction after \( 2r^2 \) turns, and so on (Figs. 5, 6). A certain remarkable pattern unwinds:

Firstly the algorithm never returns to 0, nor to any other stage. (By contrast, any division of the circle by a finite whole number ratio \( \frac{a}{b} \) will return after \( b \) steps completing \( a \) revolutions. Five \( \frac{5}{2} \) turns, for example, complete the circuit of the regular pentagram in two revolutions.) This is an expression of the irrationality of \( r \). It is a pattern which translates into a
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practicality for plants: Successive leaves or branches put out around a stem are spaced for sun and growing room.

Secondly, every step of the algorithm cuts an existing part of the circumference further into a golden ratio of parts.

Thirdly, every stage approximates to a balanced division of the whole circle. Again, this translates into a practical efficiency for plants in their relation to sun, space and gravity. (There is of course no angle which if used in such an algorithm could lead to a division into any number of equal parts.) At any stage the whole circumference is divided into a pattern of just three sizes (and periodically, just two) of arc: \( r^n \) and \( r^{n+1} \) after \( f_{n+1} - 1 \) steps; \( r^n \), \( r^{n+1} \) and \( r^{n+2} \) after \( x \) steps where \( f_{n+2} \leq x < f_{n+3} - 1 \).

Every step further divides only the largest arcs. Such a pattern of successive and even division of the circle provides a key to model-making for spiral and helical phyllotaxis—the spacing that plants exhibit in their arrangement of leaves, twigs, petals, seeds, florets around a stem or centre: rose, lotus or thistle alike.

Fourthly, although the algorithm never returns to exactly the same position on the circumference, the successively nearer and nearer approaches are made on alternating sides after 1 step, 2 steps, 3 steps, 5 steps, 8 steps, .... The Fibonacci numbers!

Fifthly, the successive approaches regularly diminish in the golden ratio. This pattern expresses the relation between \( r \) and the Fibonacci ratios. The ratios 0, 1, 1, 2, 3, .... are successively closer approximations to the value of \( r \). Indeed \( \frac{f_{n+1}}{f_n} \) is a smooth succession of rational approximations. The hidden thesis of phyllotaxis is that no other number is so provisioned [7].

Sixthly, this is a regular pattern of nesting. Position 21 lies between 0 and 8. Generally, \( x \) lies between \( x - f_n \) and \( x - f_{n+1} \) for \( f_n \leq x < f_{n+1} \). This is worth bearing in mind when considering phyllotaxis pragmatically. “Between” is easier to find than a precise angle.

All of this suggests a simpler programme, number 2, which was drawn (Fig. 7). On a single slow logarithmic spiral, rhombi are drawn at regular intervals of the golden angle, and grow

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**Fig. 5.** Steps in the golden algorithm of the circle. (a) One step round the circle with the golden angle, \( 137.50776 \ldots \), obtained by dividing the circumference into a golden ratio of parts. (b) Two steps. (c) Three steps. (d) Steps four and five.

**Fig. 6.** Steps in the golden algorithm of the circle (continued). 21 steps, showing the Fibonacci pattern of successively closer returns to the starting point and the on-going pattern of gradual and balanced division of the circle.

**Fig. 7.** Computer graphic (programme 2), achieving a pattern equivalent to Fig. 3 by the use of a single spiral on which the rhombi are placed at intervals of the golden angle.
uniformly in proportion with the host spiral. Fibonacci spirals result from one spiral! The growth, however, is uniform.

The third programme was devised to overcome this limitation (Fig. 8). The following function of a function is used as a model of the growth of the seeds: the \( s_i = a \arctan (bc^i) \) with suitably chosen constants \( a, b \), and \( c \). This gives a growth pattern which, beginning exponentially, slows to a limit. These seeds are then placed on a single host spiral which sweeps out in step with the accumulating areas of seeds \( r_i^2 - r_{i-1}^2 = k s_i^2 \) and \( \theta_i - \theta_{i-1} = r^2 \) turns. Making \( r_0 = s_0 = a \arctan (b) \) initiates the sequence of packing.

Now the resultant Fibonacci spirals of contacting seeds change through the ratios as the seeds slow and cease growing! But equally striking is the continuity of all the spirals, as they emerge exponentially and decay into a form which is tighter than the Archimedean spiral, i.e. \( r^2 = K \theta \). The green spiral makes a space-filling array out of a single non-uniformly growing sequence of seeds.

It is perhaps important to bear in mind the distinction between the way in which a plant would grow into such a pattern and the way the computer builds it by addition of static parts. So far the mathematics describes only a static condition, a momentary form in a pattern that grows. Plants grow by a succession of new members added at the centre and continued enlargement.

The growth pattern \( s_i = a \arctan (bc^i) \) may be given an infinite number of interpretations by alteration of constants \( a, b \), and \( c \). This would provide not only a sequence of forms through which an individual plant might grow in time, but also different sequences for different species, defined morphologically.

The limitations of working only in the plane and the particularity of the growth function are to be reviewed.

As a contrasting experiment it was decided to repeat this programme but using irrational angles other than \( r^2 \) (Fig. 9). Four irrationals were chosen to divide the circle, the angles being \( \frac{1}{\pi}, \frac{1}{e}, \frac{1}{\sqrt{5}} \), and \( \frac{2}{3} \). A failure to close-pack is observed. The more scattered patterns of interlocking spirals are interpreted as before, as indicating the succession of rational approximations.

Any rational division of the circle, say \( \frac{a}{b} \), would in our algorithm lead to straight rays: \( b \) of them, leaving ever-increasing gaps at the circumference. An irrational number which is close in value to \( \frac{a}{b} \) would approximate to the behaviour of \( \frac{a}{b} \) but gain or lose a small increment each revolution: spirals! Figure 9a shows \( \frac{1}{\pi} \) which approximates to \( \frac{1}{4} \) and then to \( \frac{22}{7} \).

Hence the appearance of three spirals, then twenty-two.

The switch-over of spirals from the denominator of one approximation to the denominator of the next approximation is an expression of the increasing number of elements needed to fill a growing circumference. The “gappiness” of the patterns expresses the gaps in the sequence of positive whole numbers between denominators of successive rational approximations.

Any irrational number, say \( j \), would have a “signature” in the finite whole numbers: a succession of denominators; \( f^{-1} \) would give the succession of numerators of \( j \) as denominators.

The study so far thus points in three directions: to the world of plants, from the skewed tessellation patterns of fossilized tree barks to the cauliflower whose every element is another pattern within the pattern; to number theory; and to computerised art.

I wish to express my thanks to Professor H. S. M. Coxeter, Cliff Edwards, Dr. Holiday, Keith Laws, Alan Senior and Rose Spilberg for inspiring and assisting this study. All computer drawings were drawn by Keith Laws, whose computer graphics system was developed with Ensor Holiday for the Altair Project.
REFERENCES AND NOTES

1. In the case of 12 circles, $\sqrt{3} - 1 : 2\sqrt{2}$.
2. In the case of 12 circles, $1.582 \ldots : 1$.
3. Alan Senior pointed this out to me, after which we pursued the problem jointly.
5. Cliff Edwards pointed this out to us.
6. "r" is the notation used by H. S. M. Coxeter in his Introduction to Geometry (New York: Wiley, 1961), also by Fejes Tóth, Tutte and many others. Some writers on the subject use "ϕ".

[Editor's note: A number of articles dealing with the golden section have been published in Leonardo. One concerned with biology is: David S. Fensom, The Golden Section and Human Evolution, Leonardo 14, 232 (1981).]