# Random variables & probability models

## Objective

(1) Learn the concept and meaning of random variables.

(2) Learn how to define random variables & construct probability models.

## Concept briefs:

* Random variable: A variable whose possible values are numerical outcomes of some random process. Example: Let X=the number of heads in 5 flips of a coin. Then the possible values of X are {0, 1, 2, 3, 4, 5}.
* Probability model: For a random variable X, the list of probabilities associated with each possible value $x$ is its probability model. It is denoted $P(x)$.
* Expected value & SD: The expected value of a random variable X is like its mean value. It is given by E(X)= $\overline{x}=\sum x \cdot P(x)$. The standard deviation of X is SD(X) = $\sqrt{\sum(x-\overline{x})^2 P(x)}$.
* Continuous random variable: Some random processes have infinite possible outcomes, continuously distributed across some numerical range. A random variable associated with such a process is a continuous random variable. Example: The local temperature recorded by a thermometer during a 24-hour period.
* Effect of arithmetic operations

# Random variables & expected values

## A Warmup

Let's (once again!) toss a fair coin twice and see what we can learn. This time we want to know how many heads we can expect, on average.

(1) As a first step, what does your intuition say?

(2) Now, let's develop a formal process that will work even when the problem gets much more complex.

Let X be a random variable that represents the number of heads we can get in two tosses.

(i) What are all the possible values of X?

(ii) Find the probability of each value, and make a probability model for X.

(iii) Find E(X) using the probability model.

(iv) Find SD(X).

## Some answers:

The probability model is shown.

E(X) $=\sum x \cdot P(x)$

= 0*1/4 + 1*1/2 + 2*1/4

= 1

| X | P(x) |
|---|---|
| 0 | 1/4 |
| 1 | 1/2 |
| 2 | 1/4 |

SD(X) $=\sqrt{\sum (x-\overline{x})^2 P(x)}$
with $\overline{x}$= E(X)

$= \sqrt{(0-1)^2(1/4) + (1-1)^2(1/2) + (2-1)^2(1/4)} = 1/\sqrt{2}$

## Another Warmup

What is the expected value of the outcome if I roll a die once. Assume standard, 6-sided, fair die.

(1) Intuition?

(2) Formal: Let X be a random variable that represents the outcome. Make a probability model for X and compute its expected value.

E(X) = $\sum x \cdot P(x)$

= 1*1/ 6 + 2*1/6 + 3*1/6 + 4*1/ 6 + 5*1/6 + 6*1/6

= 3.5

| X | P(x) |
|---|---|
| 1 | 1/6 |
| 2 | 1/6 |
| 3 | 1/6 |
| 4 | 1/6 |
| 5 | 1/6 |
| 6 | 1/6 |

SD(X) = $\sqrt{\sum (x-\overline{x})^2 P(x)}$

with $\overline{x}$= E(X)

SD(X) = $\sqrt{(1-3.5)^2(1/6)\ +\ (2-3.5)^2(1/6)\ +\cdots+ (6-3.5)^2(1/6)}$

= 1.7078

## Example

A card game: You draw a card at random from a deck. If you get a red card, you win nothing. If you get a spade, you win $7. For any club, you win $15, plus an extra $25 for the ace of clubs.

(a) Create a probability model for the amount you win.

(b) Find the expected amount you'll win.

(c) Find the standard deviation of the amount.

**Solution**

(a) Let X=random variable that represents the amount I win.

The possible values of X are:

$40, $ 15, $7, $0

The probability model is shown in the table.

| X | P(x) |
|---|---|
| $40 | 1/52 |
| $15 | 12/52 |
| $7 | 13/52 |
| $0 | 26/52 |

Reasoning: I win $40 if I get the ace of clubs. There is only one such card in a deck of 52. Hence the probability of $40 is 1/52. There are 12 more clubs in a deck. Hence the probability of $15 is 12/52. Etc.

(b) E(X) = $\sum x \cdot P(x)$

= 40*(1/ 52) + 15*(12/52) + 7*(13/52) + 0*(26/52)

**= $5.98**

(c) SD(X) = $\sqrt{\sum(x-\overline{x})^2 P(x)}$ with $\overline{x}$= E(X)

$$= \sqrt{(40-5.98)^2\left(\frac{1}{52}\right) + (15-5.98)^2\left(\frac{12}{52}\right) + (7-5.98)^2\left(\frac{13}{52}\right) + (0-5.98)^2(26/52)}$$

$$= \sqrt{59.1727}$$

**= $7.6924**

# Effect of arithmetic operations

## What does that mean

Suppose X is some random variable (e.g., # of heads when we toss a coin twice). We know that its mean is E(X)= $\overline{x}$= $\sum x \cdot P(x)$ and SD(X)= $\sqrt{\sum (x-\overline{x})^2 P(x)}$.

Now, suppose we add a constant *c* to each value of X, what effect does it have on the new E(X) and S(X)?

Alternatively, if we multiply every value by *c*, what is the effect?

## Addition

Mean: E(X+c) = c + E(X)
Variance: Var(X+c) = Var(X)
Standard deviation: SD(X+c) = SD(X) = square root of variance

Moral: Addition affects the mean, but not the variance or SD.

## Multiplication

Mean: E(c*X) = c*E(X)
Variance: Var(c*X) = $c^2$ * Var(X)
Standard deviation: SD(c*X) = |c|*SD(X)
= square root of variance

Moral: Multiplication affects all the summary stats.

Similarly, it is sometimes necessary to add/subtract two different random numbers, say, X, Y. Here is the effect that has

Mean: E(X+Y) = E(X)+E(Y); E(X-Y) = E(X)-E(Y)

Variance: Var(X±Y) = Var(X)+Var(Y)
[note that variances always add, even for X-Y]

Standard deviation: SD(X±Y)= square root of Var(X±Y)

## Example

Let X, Y be two random variables, and suppose we are given

$$E(X)=6,\ \ SD(X)=2,\ \ E(Y)=50,\ \ SD(Y)=12$$

Find the expected value and standard deviation of each of the following:

(a) X – 10

(b) −3Y

(c) X+Y

(d) X−3Y

**Solution**

(a) $E(X-10) = E(X) - 10 = -4$

$SD(X-10) = SD(X) = 2$

(b) $E(-3Y) = -3*E(Y) = -150$

$SD(-3Y) = |-3|*SD(Y) = 36$

(c) $E(X+Y) = E(X)+E(Y) = 56$

$VAR(X+Y) = VAR(X)+VAR(Y) = 2^2 + 12^2 = 148$

$SD(X+Y) = \sqrt{VAR(X+Y)} = \sqrt{148} = 12.1655$

(d) $E(X-3Y) = E(X) - 3*E(Y) = 6 - 3*50 = -144$

$VAR(X-3Y) = VAR(X) + VAR(3Y) = 2^2 + (3*12)^2 = 4+1296$

$SD(X-3Y) = \sqrt{VAR(X-3Y)} = \sqrt{1300} = 36.0555$

## Section 14.2

3. **Oranges again** What is the standard deviation for Exercise 1?

4. **Caffeinated again** What is the standard deviation for Exercise 2?

## Section 14.3

5. **Salary** An employer pays a mean salary for a 5-day workweek of $1250 with a standard deviation of $129. On the weekends, his salary expenses have a mean of $450 with a standard deviation of $57. What is the mean and standard deviation of his total weekly salaries?

6. **Golf scores** A golfer keeps track of his score for playing nine holes of golf (half a normal golf round). His mean score is 85 with a standard deviation of 11. Assuming that the second 9 has the same mean and standard deviation, what is the mean and standard deviation of his total score if he plays a full 18 holes?

## Section 14.4

7. **Toasters** A manufacturer ships toasters in cartons of 20. In each carton, they estimate a 5% chance that one of the toasters will need to be sent back for minor repairs. What is the probability that in a carton, there will be exactly 3 toasters that need repair?

8. **Soccer** A soccer team estimates that they will score on 8% of the corner kicks. In next week's game, the team hopes to kick 15 corner kicks. What are the chances that they will score on 2 of those opportunities?

## Section 14.5

9. **Toasters again** In a batch of 10,000 toasters, what are the chances that fewer than 450 need to be returned?

10. **Soccer again** If this team has 200 corner kicks over the season, what are the chances that they score more than 22 times?

## *Section 14.6

11. **Sell!** A car dealership sells an average of 5 cars in a day. Using the Poisson model, what is the probability that the dealer sells 3 cars tomorrow?

12. **Passing on** A large hospital has an average of 7 fatalities in a week. Using the Poisson model, what is the probability that this week it has 10 fatalities?

## Section 14.7

13. **Battery** The life span of a calculator battery is normally distributed with a mean of 45 hours and a standard deviation of 5 hours. What is the probability that a battery lasts more than 53 hours?

14. **Ketchup** An automatic filling machine in a factory fills bottles of ketchup with a mean of 16.1 oz and a standard deviation of 0.05 oz with a distribution that can be well modeled by a Normal model. What is the probability that your bottle of ketchup contains less than 16 oz?

## Chapter Exercises

15. **Expected value** Find the expected value of each random variable:

a)

| $x$ | 10 | 20 | 30 |
|---|---|---|---|
| $P(X = x)$ | 0.3 | 0.5 | 0.2 |

b)

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $P(X = x)$ | 0.3 | 0.4 | 0.2 | 0.1 |

16. **Expected value** Find the expected value of each random variable:

a)

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(X = x)$ | 0.2 | 0.4 | 0.4 |

b)

| $x$ | 100 | 200 | 300 | 400 |
|---|---|---|---|---|
| $P(X = x)$ | 0.1 | 0.2 | 0.5 | 0.2 |

17. **Pick a card, any card** You draw a card from a deck. If you get a red card, you win nothing. If you get a spade, you win $5. For any club, you win $10 plus an extra $20 for the ace of clubs.
   a) Create a probability model for the amount you win.
   b) Find the expected amount you'll win.
   c) What would you be willing to pay to play this game?

18. **You bet!** You roll a fair die. If it comes up a 6, you win $100. If not, you get to roll again. If you get a 6 the second time, you win $50. If not, you lose and win nothing.
   a) Create a probability model for the amount you win.
   b) Find the expected amount you'll win.
   c) What would you be willing to pay to play this game?

19. **Kids** A couple plans to have children until they get a girl, but they agree that they will not have more than three children even if all are boys. (Assume boys and girls are equally likely.)
   a) Create a probability model for the number of children they might have.
   b) Find the expected number of children.
   c) Find the expected number of boys they'll have.

20. **Carnival** A carnival game offers a $100 cash prize for anyone who can break a balloon by throwing a dart at it. It costs $5 to play (they keep your $5, even if you win the $100), and you're willing to spend up to $20 trying to win. You estimate that you have about a 10% chance of hitting the balloon on any throw.
   a) Create a probability model for this carnival game.
   b) Find the expected number of darts you'll throw.
   c) Find your expected winnings.

21. **Software** A small software company bids on two contracts and knows it can only get one of them. It

anticipates a profit of $50,000 if it gets the larger contract and a profit of $20,000 on the smaller contract. The company estimates there's a 30% chance it will get the larger contract and a 60% chance it will get the smaller contract. Assuming the contracts will be awarded independently, what's the expected profit?

22. **Racehorse** A man buys a racehorse for $20,000 and enters it in two races. He plans to sell the horse afterward, hoping to make a profit. If the horse wins both races, its value will jump to $100,000. If it wins one of the races, it will be worth $50,000. If it loses both races, it will be worth only $10,000. The man believes there's a 20% chance that the horse will win the first race and a 30% chance it will win the second one. Assuming that the two races are independent events, find the man's expected profit.

23. **Variation 1** Find the standard deviations of the random variables in Exercise 15.

24. **Variation 2** Find the standard deviations of the random variables in Exercise 16.

25. **Pick another card** Find the standard deviation of the amount you might win drawing a card in Exercise 17.

26. **The die** Find the standard deviation of the amount you might win rolling a die in Exercise 18.

27. **Kids again** Find the standard deviation of the number of children the couple in Exercise 19 may have.

28. **Darts** Find the standard deviation of your winnings throwing darts in Exercise 20.

29. **Repairs** The probability model below describes the number of repair calls that an appliance repair shop may receive during an hour.

| **Repair Calls** | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| **Probability** | 0.1 | 0.3 | 0.4 | 0.2 |

a) How many calls should the shop expect per hour?
b) What is the standard deviation?

30. **Red lights** A commuter must pass through five traffic lights on her way to work and will have to stop at each one that is red. She estimates the probability model for the number of red lights she hits, as shown below.

| $X$ = # of Red | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P(X = x)$ | 0.05 | 0.25 | 0.35 | 0.15 | 0.15 | 0.05 |

a) How many red lights should she expect to hit each day?
b) What's the standard deviation?

31. **Defects** A consumer organization inspecting new cars found that many had appearance defects (dents, scratches, paint chips, etc.). While none had more than three of these defects, 7% had exactly three, 11% exactly two, and 21% only one defect. Find the expected number of appearance defects in a new car and the standard deviation.

32. **Insurance** An insurance policy costs $100 and will pay policyholders $10,000 if they suffer a major injury (resulting in hospitalization) or $3000 if they suffer a minor injury (resulting in lost time from work). The company estimates that each year 1 in every 2000 policyholders may have a major injury, and 1 in 500 a minor injury only.
a) Create a probability model for the profit on a policy.
b) What's the company's expected profit on this policy?
c) What's the standard deviation?

33. **Cancelled flights** Mary is deciding whether to book the cheaper flight home from college after her final exams, but she's unsure when her last exam will be. She thinks there is only a 20% chance that the exam will be scheduled after the last day she can get a seat on the cheaper flight. If it is and she has to cancel the flight, she will lose $150. If she can take the cheaper flight, she will save $100.
a) If she books the cheaper flight, what can she expect to gain, on average?
b) What is the standard deviation?

34. **Day trading** An option to buy a stock is priced at $200. If the stock closes above 30 on May 15, the option will be worth $1000. If it closes below 20, the option will be worth nothing, and if it closes between 20 and 30 (inclusively), the option will be worth $200. A trader thinks there is a 50% chance that the stock will close in the 20–30 range, a 20% chance that it will close above 30, and a 30% chance that it will fall below 20 on May 15.
a) How much does she expect to gain?
b) What is the standard deviation of her gain?
c) Should she buy the stock option?

35. **Contest** You play two games against the same opponent. The probability you win the first game is 0.4. If you win the first game, the probability you also win the second is 0.2. If you lose the first game, the probability that you win the second is 0.3.
a) Are the two games independent? Explain.
b) What's the probability you lose both games?
c) What's the probability you win both games?
d) Let random variable $X$ be the number of games you win. Find the probability model for $X$.
e) What are the expected value and standard deviation?

36. **Contracts** Your company bids for two contracts. You believe the probability you get contract #1 is 0.8. If you get contract #1, the probability you also get contract #2 will be 0.2, and if you do not get #1, the probability you get #2 will be 0.3.
a) Are the two contracts independent? Explain.
b) Find the probability you get both contracts.
c) Find the probability you get no contract.
d) Let $X$ be the number of contracts you get. Find the probability model for $X$.
e) Find the expected value and standard deviation.

37. **Batteries** In a group of 10 batteries, 3 are dead. You choose 2 batteries at random.
a) Create a probability model for the number of good batteries you get.

b) What's the expected number of good ones you get?
c) What's the standard deviation?

**38. Kittens** In a litter of seven kittens, three are female. You pick two kittens at random.

a) Create a probability model for the number of male kittens you get.
b) What's the expected number of males?
c) What's the standard deviation?

**39. Random variables** Given independent random variables with means and standard deviations as shown, find the mean and standard deviation of:

a) $3X$
b) $Y + 6$
c) $X + Y$
d) $X - Y$
e) $X_1 + X_2$

| | Mean | SD |
|---|---|---|
| X | 10 | 2 |
| Y | 20 | 5 |

**40. Random variables** Given independent random variables with means and standard deviations as shown, find the mean and standard deviation of:

a) $X - 20$
b) $0.5Y$
c) $X + Y$
d) $X - Y$
e) $Y_1 + Y_2$

| | Mean | SD |
|---|---|---|
| X | 80 | 12 |
| Y | 12 | 3 |

**41. Random variables** Given independent random variables with means and standard deviations as shown, find the mean and standard deviation of:

a) $0.8Y$
b) $2X - 100$
c) $X + 2Y$
d) $3X - Y$
e) $Y_1 + Y_2 + Y_3 + Y_4$

| | Mean | SD |
|---|---|---|
| X | 120 | 12 |
| Y | 300 | 16 |

**42. Random variables** Given independent random variables with means and standard deviations as shown, find the mean and standard deviation of:

a) $2Y + 20$
b) $3X$
c) $0.25X + Y$
d) $X - 5Y$
e) $X_1 + X_2 + X_3$

| | Mean | SD |
|---|---|---|
| X | 80 | 12 |
| Y | 12 | 3 |

**43. Eggs** A grocery supplier believes that in a dozen eggs, the mean number of broken ones is 0.6 with a standard deviation of 0.5 eggs. You buy 3 dozen eggs without checking them.

a) How many broken eggs do you expect to get?
b) What's the standard deviation?
c) What assumptions did you have to make about the eggs in order to answer this question? Do you think that assumption is warranted? Explain.

**44. Garden** A company selling vegetable seeds in packets of 20 estimates that the mean number of seeds that will actually grow is 18, with a standard deviation of 1.2 seeds. You buy 5 different seed packets.

a) How many bad (non-growing) seeds do you expect to get?
b) What's the standard deviation of the number of bad seeds?
c) What assumptions did you make about the seeds? Do you think that assumption is warranted? Explain.

**45. Repair calls** Suppose that the appliance shop in Exercise 29 plans an 8-hour day.

a) Find the mean and standard deviation of the number of repair calls they should expect in a day.
b) What assumption did you make about the repair calls?
c) Use the mean and standard deviation to describe what a typical 8-hour day will be like.
d) At the end of a day, a worker comments "Boy, I'm tired. Today was sure unusually busy!" How many repair calls would justify such an observation.

**46. Stop!** Suppose the commuter in Exercise 30 has a 5-day workweek.

a) Find the mean and standard deviation of the number of red lights the commuter should expect to hit in her week.
b) What assumption did you make about the days?
c) Use the mean and standard deviation to describe a typical week.
d) Upon arriving home on Friday, the commuter remarks, "Wow! My commute was quick all week." How many red lights would it take to deserve a feeling of good luck?

**47. Tickets** A delivery company's trucks occasionally get parking tickets, and based on past experience, the company plans that the trucks will average 1.3 tickets a month, with a standard deviation of 0.7 tickets.

a) If they have 18 trucks, what are the mean and standard deviation of the total number of parking tickets the company will have to pay this month?
b) What assumption did you make in answering?
c) What would be an unusually bad month for the company?

**48. Donations** Organizers of a televised fundraiser know from past experience that most people donate small amounts ($10–$25), some donate larger amounts ($50–$100), and a few people make very generous donations of $250, $500, or more. Historically, pledges average about $32 with a standard deviation of $54.

a) If 120 people call in pledges, what are the mean and standard deviation of the total amount raised?
b) What assumption did you make in answering the question in part a?
c) The organizers are worried that people are pledging less than normal. How low would the 120-person total need to be to convince you that this is the case?

**49. Fire!** An insurance company estimates that it should make an annual profit of $150 on each homeowner's policy written, with a standard deviation of $6000.

a) Why is the standard deviation so large?
b) If it writes only two of these policies, what are the mean and standard deviation of the annual profit?
c) If it writes 10,000 of these policies, what are the mean and standard deviation of the annual profit?