Probability basics

Objective

(1) Understand the concept of statistical probability.

(2) Learn how to compute probabilities for "simple" events.

Concept briefs:

* Trial, outcome, event - what is the difference?

* Probability = Relative frequency with which an event occurs. If all outcomes are equally likely, probability is simply the proportion of favorable outcomes.

* Disjoint events = They have no outcomes in common.

* Independent events = One's outcome has no effect on other's outcome.

* Complement rule: P(A) = 1 – P(not A).

* For disjoint events: P(A or B) = P(A) + P(B).

* For independent events: P(A and B) = P(A) * P(B).

Warmup questions:

* If I toss a coin once, what is the probability I will get heads.

* If I toss a coin twice, what is the probability of both heads?

* If I toss a coin 3 times, what is the probability all 3 heads?

Cheating dice again:

* Suppose I custom-make a “die” that has 6’s on three sides, and the remaining three sides have 3, 4 & 5. If I roll this “die” and look at the number I get:

  - What is the probability of NOT getting a 6?
  - What is the probability of getting an even number?

Coin Toss Outcomes:

	1 Toss	2 Tosses	3 Tosses
All possible outcomes	H T Total=2	H H H T T H TT Total=4	H H H H H T H T H H T T T H H T H T T T H T T T Total=8

Die roll outcomes: 3, 4, 5, 6, 6, 6

Moral of the story: Make a list of all possible outcomes. Look at what proportion of these gives the desired outcomes, to get the probability.

Trial, outcome, event - How are they different?

Illustration: Toss a coin 2 times

Trial: Each sequence of 2 tosses.

Outcome: There are 4 possible outcomes

{H, H}, {T, H}, {H, T}, {T, T}

Sample space: Collection of all possible outcomes.

Event: Can define any individual outcome, or group of outcomes as an event.

E.g., Event A = Both heads ---> {H, H} is only acceptable outcome.

Event B = At least 1 head ---> {H, H}, {T, H}, {H, T} are acceptable.

Illustration 2: Roll 2 dice

Trial: Each roll of the 2 dice.

Outcome: There are 36 possible outcomes [Exercise: Name them.]

e.g., {1, 1}, {1, 2}, {1, 3}, **. . .**, {2, 1}, {2, 2}, etc.

Sample space: Collection of all 36 outcomes.

Event: Can define any individual outcome, or group of outcomes as an event.

E.g., Event A = The sum is 2 ---> {1, 1} is only acceptable outcome.

Event B = Both dice come up even ---> Several acceptable outcomes. Can you name them?

Event C = The average is 6 [this is the same as "sum is 12." Why?]

A fundamental probability concept

If all outcomes are equally likely, then:

This depends on the specific event of interest

$$\text{Probability} = \frac{\text{\# of favorable outcomes}}{\text{Total \# of outcomes}}$$

This does not depend upon event

Warmup Exercises

(I) For 2 coin tosses, find the probability of each of the following events:

A = at least 1 head
B = first heads, then tails
C = not B

(II) For 3 coin tosses, find each of the following probabilities:

A = at least 1 head
B = at least 2 heads
C = not B
D = C and A

Key definitions & background concepts

Probability: For events with random individual outcomes, probability is the *long-term relative frequency* of a specific outcome.

Disjoint events: Two (or more) events are disjoint if they are mutually exclusive, i.e., they have no outcomes in common.

E.g., Toss a coin twice: A = both heads, B = at least 1 tail.

Independent events: Two (or more) events are independent if the outcome of one has no effect on the outcome of the other.
E.g., Toss a coin twice: A = first toss gives heads, B = 2nd toss gives tails.

--> It follows that if two events are disjoint, they cannot be independent!
[Q: Explain why, with example]

Five basic rules

(1) Probability is always between 0 and 1.

Q: What does P=0 or P=1 mean?

Q2: Give examples of P=0 and P=1 events.

(2) Collective probability of all possible outcomes is always 1.

E.g.: P(heads) + P(tails) = 1 for a coin toss

E.g.: P(hearts) + P(diamonds) + P(spades) + P(clubs) = 1 in a deck of 52 cards.

(3) Probability of A = 1 – probability of NOT-A.

E.g.: P(hearts) = 1 – probability of not getting hearts, in deck of 52 cards.

(4) Simple Addition Rule: For disjoint events, the probability that one or the other occurs is the sum of the individual probabilities.

E.g.: Probability of getting hearts OR spades = P(hearts) + P(spades)

(5) Simple Multiplication Rule: For independent events, the probability that both will occur is the product of the individual probabilities.

E.g.: If I pull one card each from 2 decks of 52 cards, the probability of getting 1 heart and 1 spade = P(hearts) * P(spades)

Conditional probability & more

Objective

(1) Learn how to work with conditional probabilities.

(2) Learn graphical techniques used in computing probabilities.

(3) Understand how "drawing without replacement" affects probability.

Concept briefs:

* Conditional probability background: For 2 non-independent events A and B, if one has already occurred, what's the probability other also occurred? e.g., If you know A has occurred, then P(B) should go up (or down).

* Conditional probability concept: Frequency of an event (say, B) relative to another event (say, A) $\Rightarrow$ proportion of common outcomes of B within A.

* Conditional probability techniques: (1) Two-way tables, (2) Venn diagrams, (3) Tree diagrams.

* General addition rule: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

* General multiplication rule: $P(A \text{ and } B) = P(A) \times P(B \mid A) = P(B) \times P(A \mid B)$

* Independent events - Technical method: If $P(B \mid A) = P(B)$, then A and B are independent.

* Drawing w/o replacement: Some probabilities are affected by sample size. E.g., What is probability of drawing 4 back-to-back hearts from 52 cards? Would this change if we put the card back into the deck after each draw?

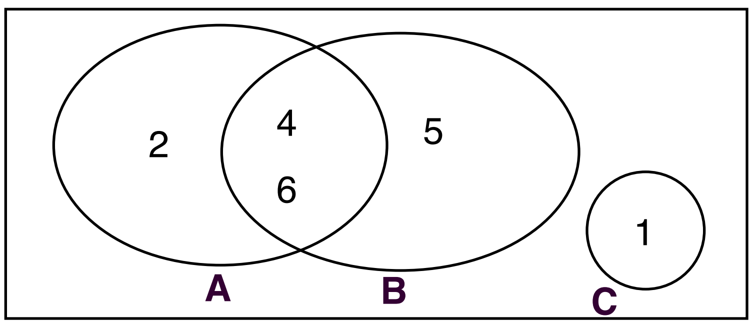
Conditional probability with Venn diagrams

Illustration: Roll a die once, look at outcome.

Event A = Even number $\Rightarrow \{2, 4, 6\}$

Event B = Number larger than 3 $\Rightarrow \{4, 5, 6\}$

Event C = The number 1 $\Rightarrow \{1\}$



<u>NOTICE</u> that A and B aren't disjoint

but C is disjoint from both.

Q: What is P(B|A)? [i.e., probability of B, given A]

$\Rightarrow$ Look for relative frequency of B within A.

$$= \frac{\text{\# of outcomes of B within A}}{\text{\# of outcomes in A}} = \frac{2}{3}$$

Q2: What is P(A|B)? [i.e., probability of A, given B]

Here it is 2/3 also.

<u>BUT</u>, be aware it is not always the same as P(B|A).

Q3: What is P(C|A)?!

General addition rule

Recall: Simple addition rule, P(A or B) = P(A) + P(B), requires disjoint events.

General rule for any 2 events is:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

See previous Venn diagram: Key idea here is to subtract out the double count of P(A and B) that happens if we simply do P(A) + P(B).

General multiplication rule

Recall: Simple multiplication rule, P(A and B) = P(A) x P(B), requires independent events.

General rule for any 2 events is:

$$P(A \text{ and } B) = P(A) \times P(B|A) \quad \textbf{OR} \quad P(A \text{ and } B) = P(B) \times P(A|B)$$

NOTES:

* This rule is used in different ways.

* As written, it gives us a way to compute P(A and B).

* But, often used to find conditional probabilities by rewriting as

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} \quad \text{OR} \quad P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Independence - Technical definition

To prove independence of A and B, you must compute & show that

$$P(B|A) = P(B) \quad \text{OR} \quad P(A|B) = P(A)$$

In words:

"The probability of B remains unchanged even if A has occurred."

OR

"The probability of A remains unchanged even if B has occurred."

Common errors to watch for

(1) Misinterpreting conditional probability wording to mean "and" probability.

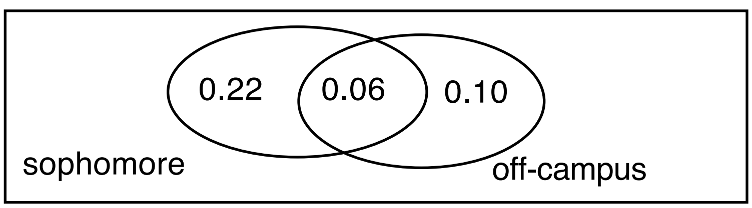
E.g., (a) Find the probability that a natural sciences student is also a smoker.
(b) Find the probability that a student is in the natural sciences and smokes.
[Can you tell which is which kind of probability?]

(2) Misinterpreting "or" probabilities as mutually exclusive.

E.g., What is the probability that a student is a sophomore or lives off campus. [This should include sophomores, off campus residents, as well as those who are both.]

(3) When using Venn diagrams to compute conditional probability, students sometimes forget to include the intersection region in the calculations.

E.g., Suppose you're given: P(sophomore) = 0.28, P(off-campus) = 0.16,
P(sophomore and off-campus) = 0.06.



Then, it follows that: P(sophomore | off-campus) = 0.06 / (0.06 + 0.10)
P(sophomore OR off-campus) = 0.22 + 0.06 + 0.10

(4) Forgetting to draw without replacement when sample size is small.

E.g., In a class of 20 students, suppose there are 6 liberals (30%), 10 moderates (50%), and 4 conservatives (20%). The probability of randomly picking 3 conservatives back-to-back is not .2x.2x.2. It is .2x(3/19)x(2/18).

(5) Forgetting to verify "independent" or "disjoint" before using simple multiplication or addition rule.

E.g., Suppose P(C)=0.4, P(D)=0.7, P(C|D)=0.6.
Then P(C and D) is NOT 0.4x0.7. P(C or D) is NOT 0.4+0.7.