Quiz: September 13

This is a closed-book quiz, and no team-work or reference materials are permitted.

1. Give a mathematically precise definition of image and pre-image (in connection with functions). Be sure to include any context needed for your definition to make sense.

   Example of needed context: If you are trying to define "relation," be sure to indicate the sets that your relation will try to relate. Example of a complete definition of relation: "A relation from set A to set B is any subest of $A \times B$, where $A \times B = \{(m, n) \mid m \in A \text{ and } n \in B\}$"

2. Let $g : \mathbb{Z} \to \mathbb{Z}$ be a function defined by

$$g(x) = \begin{cases} x + 10 & \text{if } x \text{ is odd} \\ x - 11 & \text{if } x \text{ is even} \end{cases}$$

   Let $S = \{x \in \mathbb{N} \mid 2 \leq x < 8\}$. Find $g^{-1}(S)$ and $g(g^{-1}(S))$.

Solution

1. Let A and B be sets and $f : A \to B$ be a function.

   The image of any $C \subseteq A$ is defined by

$$f(C) = \{y \in B \mid y = f(x) \text{ for some } x \in C\}$$

OR

$$f(C) = \{y \in B \mid \text{there exists } x \in C \text{ for which } y = f(x)\}$$

   The pre-image of any $D \subseteq B$ is defined by

$$f^{-1}(D) = \{x \in A \mid f(x) \in D\}$$

2. We have $S = \{2, 3, 4, 5, 6, 7\}$.
   To find $g^{-1}(S)$: We notice that the range of $g(x)$ only contains odd numbers. Thus the even numbers in S have no inverse image. That means we only need to consider the odd numbers in S.

   For example, if $g(x) = 3$ then $x = -7$ and $x = 14$ are both valid since $g(-7) = g(14) = 3$.

   In this way, we get: $\boxed{g^{-1}(S) = \{-7, 14, -5, 16, -3, 18\} = \{-7, -5, -3, 14, 16, 18\}}$

   For $g(g^{-1}(S))$, we must find $g(\{-7, 14, -5, 16, -3, 18\})$.
   Plugging these elements into $g(x)$ gives: $g(-7) = g(14) = 3$, etc.

   Therefore, $\boxed{g(g^{-1}(S)) = \{3, 5, 7\}}$

Grading: Total points possible = 5.

2.5 pt for (1): 0.5 pt=clarify sets A, B, $f : A \to B$; 1+1 pt = each correct defn.

2.5 pt for (2): 1 pt for each correct answer; 0.5 pt for steps/reasons.