Student name:

MATH 288: Intro to Proof | **Midterm Test**
Fall 2021 | **August. 27, 2021**

Instructions:

- Answer all questions on separate paper (not on this sheet!).
- This is a regular "closed-book" test, and is to be taken without the use of notes, books, or other reference materials.
- This test contains questions numbered (1) to (7). It adds up to 34 points.

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(1) [2 pts. each$\times 3 = 6$ pts.] Negate the following statements. Negation must be written in complete words and sentences, with sparing use of symbolism. (Yes, you may use the standard arithmetic symbols used in the questions themselves –, =, $\leq$, etc.)

(a) There exists a natural number m such that for every integer n, $m-n$ is a natural number.

(b) If f is a continuous function on $\mathbb{R}$, then $f(a)$ is defined for every $a \in \mathbb{R}$.

(c) For each $\epsilon > 0$, there exists a real number N such that $S_n < \epsilon$ whenever $n > N$.

(2) [3 pts. each$\times 2 = 6$ pts.] Consider the following pairs of sets

(a) $S = \{x \in \mathbb{N} \mid x > 8\}$ and $T = \{x \in \mathbb{N} \mid x \leq 14\}$

(b) $S = \{x \in \mathbb{R} \mid -3 < x \leq 3\}$ and $T = \{x \in \mathbb{R} \mid x < 10\}$

Find $S \cup T$ and $S \cap T$ for each pair. Show reasoning.

(3) [2 pts. each$\times 2 = 4$ pts.] Write the contrapositive of the following implications. Again, use complete words and sentences, with sparing use of symbols beyond those used in the questions.

(a) If n is a natural number, then either $\sqrt{n}$ is also a natural number or it is an irrational number.

(b) If $(A \times B) \subseteq (A \times C)$ and $B \subseteq C$, then $(A \cup B) \subseteq (A \cup C)$.

(4) [2 pts. each$\times 2 = 4$ pts.] Write the converse of the following implications.

(a) If x is a natural number, then for all natural numbers y, $x \neq 2y$.

(b) For all sets A, B and C, if $(A \cup C) \subseteq (B \cup C)$, then $A \subseteq B$.

(5) [2 pts. each$\times 2 = 4$ pts.] Assuming x, y and z are real numbers, indicate true or false for each the following, with reasons.

(a) $\forall x$ and $\forall y, \exists z$ such that $z = x - y$.

(b) $\exists x$ such that $\forall y$ and $\forall z$, $x + y = z$.

(6) [5 pts.] Let A, B and C be sets. Prove that if $A \cup B = C$ and $A \cap B = \emptyset$, then $A = C - B$.

(7) [5 pts.] Prove that for all sets A and B, $A \subseteq B$ if and only if $B^c \subseteq A^c$.

End of test

Intro to Proof: Fall 2021: Midterm solutions

(1) Negate the following statements:

(a) There exists a natural number m such that for every integer n, $m-n$ is a natural number.

Solution: For all natural numbers m there exists an integer n such that $m-n$ is not a natural number.

(b) If f is a continuous function on $\mathbb{R}$, then $f(a)$ is defined for every $a \in \mathbb{R}$.

Solution: There exists a continuous function f on $\mathbb{R}$ such that $f(a)$ is undefined for some $a \in \mathbb{R}$.

(c) For each $\epsilon > 0$, there exists a real number N such that $S_n < \epsilon$ whenever $n > N$.

Solution: There exists an $\epsilon > 0$ such that for every real number N there is some $n > N$ for which $S_n \geq \epsilon$.

Grade: 2 points each. General yardstick: -0.5 pt for each error, down to 0. Exception for (b): 0 credit if negation contains "if ... then" error.

(2) Find $S \cup T$ and $S \cap T$ for the pairs of sets given below. Show reasoning.

(a) $S = \{x \in \mathbb{N} \mid x > 8\}$ and $T = \{x \in \mathbb{N} \mid x \leq 14\}$

Solution: We have, $S = \{9, 10, 11, 12, \ldots\}$ and $T = \{1, 2, 3, \ldots, 14\}$.

Thus, $S \cap T = \{9, 10, 11, 12, 13, 14\}$.

And, $S \cup T = \mathbb{N}$, because S and T together include all the natural numbers.

(b) $S = \{x \in \mathbb{R} \mid -3 < x \leq 3\}$ and $T = \{x \in \mathbb{R} \mid x < 10\}$

Solution: Here $S = (-3, 3]$, an interval of real numbers that excludes the left end-point, but includes the right end-point.

T is the interval $(-\infty, 10)$, which excludes 10 itself, but includes all real numbers less than 10. We notice that $S \subseteq T$.

Thus, $S \cap T = S = (-3, 3]$ OR $\{x \in \mathbb{R} \mid -3 < x \leq 3\}$

And, $S \cup T = T = (-\infty, 10)$.

Grade: 3 points each. Generally: 1 pt = show correct interpretation of S and T; 1 pt = correct $S \cup T$; 1 pt = correct $S \cap T$.

(3) Write the contrapositive of the following implications:

(a) If n is a natural number, then either $\sqrt{n}$ is also a natural number or it is an irrational number.

Solution: If $\sqrt{n}$ is not a natural number and it is rational, then n is not a natural number.

(b) If $(A \times B) \subseteq (A \times C)$ and $B \subseteq C$, then $(A \cup B) \subseteq (A \cup C)$.

Solution: If $(A \cup B) \nsubseteq (A \cup C)$, then either $(A \times B) \nsubseteq (A \times C)$ or $B \nsubseteq C$.

Grade: 2 points each. Generally: 0.5 pt = show correct understanding of the word "contrapositive"; 1.5 pt = put together a correct form of the contrapositive.

(4) Write the converse of the following implications:

(a) If x is a natural number, then for all natural numbers y, $x \neq 2y$.

Solution: If for all natural numbers y, $x \neq 2y$, then x is a natural number.

(b) For all sets A, B and C, if $(A \cup C) \subseteq (B \cup C)$, then $A \subseteq B$.
Solution: For all sets A, B and C, if $A \subseteq B$ then $(A \cup C) \subseteq (B \cup C)$.

> **Grade**: 2 points each. Generally: 0.5 pt = show correct understanding of the word "converse"; 1.5 pt = put together a correct form of the converse.

(5) Assuming x, y and z are real numbers, indicate true or false for each the following, with reasons.

(a) $\forall x$ and $\forall y, \exists z$ such that $z = x - y$.

Solution: True. The statement says that for all real numbers x and y, their difference $(x - y)$ is a real number, which is true.

(b) $\exists x$ such that $\forall y$ and $\forall z,\ x + y = z$.

Solution: False. The statement says there is a single/unique value of x that will satisfy $x + y = z$ for all real numbers y and z. As a counterexample, pick two cases: $y = z = 1$, which gives $x = z - y = 0$. Next, pick $y = 1, z = 2$, which gives $x = z - y = 1$.

> **Grade**: 2 points each. Generally: 1 pt = correct answer; 1 pt = correct reason.

(6) Let A, B and C be sets. Prove that if $A \cup B = C$ and $A \cap B = \emptyset$, then $A = C - B$.
Solution:
(1) I will prove this directly.
(2) Let A, B and C be sets such that $A \cup B = C$ and $A \cap B = \emptyset$.
(3) I will prove $A = C - B$ by showing $A \subseteq C - B$ and $C - B \subseteq A$.
(4) To prove $A \subseteq C - B$:
  (4.1) Let $m \in A$.
  (4.2) Then $m \in A \cup B$. [definition of union]
  (4.3) Then $m \in C$. [since $A \cup B = C$ by hypothesis]
  (4.4) Also, $m \in A \Rightarrow m \notin B$. [since $A \cap B = \emptyset$ by hypothesis]
  (4.5) Lines (4.3) and (4.4) imply $m \in C$ and $m \notin B$. Thus $m \in C - B$. [definition of complement]
  (4.6) Lines (4.1) and (4.5) show $m \in A \Rightarrow m \in C - B$.
  Therefore, $A \subseteq C - B$ [definition of subset]
(5) To prove $C - B \subseteq A$:
  (5.1) Let $p \in C - B$.
  (5.2) Then $p \in C$ and $p \notin B$. [definition of complement]
  (5.3) $p \in C \Rightarrow p \in A \cup B$. [since $C = A \cup B$ by hypothesis]
  (5.4) $p \in A \cup B \Rightarrow p \in A$ or $p \in B$. [definition of union]
  (5.5) Line (5.2) says $p \notin B$. So the only choice in (5.4) is $p \in A$.
  (5.6) Lines (5.1) and (5.5) show $p \in C - B \Rightarrow p \in A$.
  Therefore, $C - B \subseteq A$ [definition of subset]
(6) It follows from (4) and (5) that $A = C - B$. [definition of set equality]

> **Grade**: 5 points. Distribution: 0.5pt = attempt to prove $A \subseteq C - B$;
> 1 pt = correct opening lines; 1 pt = correct remainder of proof.
> 0.5pt = attempt to prove $(C - B) \subseteq A$; 1pt + 1pt, similar to previous part.

(7) Prove that for all sets A and B, $A \subseteq B$ if and only if $B^c \subseteq A^c$.

Solution:

(1) I will prove this directly. Since it is an "if and only if" claim, the implication must be proved in both directions.

(2) Proof of $A \subseteq B \Rightarrow B^c \subseteq A^c$.

(2.1) Let $A \subseteq B$.
(2.2) To prove $B^c \subseteq A^c$, suppose $x \in B^c$.
(2.3) Then $x \notin B$. [definition of complement]
(2.4) This implies $x \notin A$, since $A \subseteq B$. [contrapositive of subset definition]
(2.5) Thus $x \in A^c$. [definition of complement]
(2.6) From lines (2.2) and (2.5), $x \in B^c \Rightarrow x \in A^c$.
(2.7) It follows that $B^c \subseteq A^c$.

(3) Proof of converse: $B^c \subseteq A^c \Rightarrow A \subseteq B$:

(3.1) Let $B^c \subseteq A^c$.
(3.2) To prove $A \subseteq B$, suppose $y \in A$.
(3.3) Then $y \notin A^c$. [contrapositive of complement definition]
(3.4) This implies $y \notin B^c$, since $B^c \subseteq A^c$. [contrapositive of subset definition]
(3.5) Thus $y \in B$. [contrapositive of complement definition]
(3.6) From lines (3.2) and (3.5), $y \in A \Rightarrow y \in B$.
(3.7) It follows that $A \subseteq B$.

(4) The proofs in (2) and (3) show that $A \subseteq B$ if and only if $B^c \subseteq A^c$.

Grade: 5 points. Distribution: 0.5pt = attempt to prove $A \subseteq B \Rightarrow B^c \subseteq A^c$;
1 pt = correct opening lines; 1 pt = correct remainder of proof.
0.5pt = attempt to prove $B^c \subseteq A^c \Rightarrow A \subseteq B$; ; 1pt + 1pt, like previous part.