# Homework due date Sep. 6

Assigned exercises: 3.1: 3, 4, 6, 7, 10.
3.2: 1, 2 (questions 2, 4, 5, 8, 9 in Examples 3.6).
3.3: 2, 4. (10 problems - depending on what/how I count)
Graded exercises: 3.1: 3. 3.2: 1, 2 (numbers 2, 8). 3.3: 4.
Total (maximum) possible points = 20.
4 pt for each of 4 graded problems, plus 4 for completion of the rest.

## Exercise 3.1

(3) Prove: $A \times B = \emptyset$ iff $A = \emptyset$ or $B = \emptyset$.

**Solution**:

(a) By considering its contrapositives, this 2-way implication can be written in the following equivalent form:

$$A \times B \neq \emptyset \text{ iff } A \neq \emptyset \text{ and } B \neq \emptyset$$

(b) Direct proof of $A \times B \neq \emptyset \;\Rightarrow\; A \neq \emptyset$ and $B \neq \emptyset$

i. Let $A$ and $B$ be sets such that $A \times B \neq \emptyset$.
ii. Then there exists some $(p, q) \in A \times B$. [since $A \times B$ is not empty]
iii. That means $p \in A$ and $q \in B$. [by defn. of cross product]
iv. This implies $A \neq \emptyset$ and $B \neq \emptyset$.
v. From lines i. and iv. it follows that $A \times B \neq \emptyset \;\Rightarrow\; A \neq \emptyset$ and $B \neq \emptyset$

(c) Direct proof of converse: $A \neq \emptyset$ and $B \neq \emptyset \;\Rightarrow\; A \times B \neq \emptyset$

i. Let $A$ and $B$ be sets such that $A \neq \emptyset$ and $B \neq \emptyset$.
ii. Then there exists some $m \in A$ and some $n \in B$.
iii. This implies $(m, n) \in A \times B$. [by defn of cross product]
iv. Therefore, $A \times B \neq \emptyset$.
v. From lines i. and iv. we have: $A \neq \emptyset$ and $B \neq \emptyset \;\Rightarrow\; A \times B \neq \emptyset$

(d) From (b) and (c) it follows that $A \times B \neq \emptyset$ iff $A \neq \emptyset$ and $B \neq \emptyset$.

(e) By contraposition, this implies: $A \times B = \emptyset$ iff $A = \emptyset$ or $B = \emptyset$.

## Exercise 3.2

The following solutions are for both questions in Exercise 3.2. The question numbers refer to those in Examples 3.6 in the textbook.

(2) $\boldsymbol{r} = \{(x, y) \in \mathbb{R}^2 \mid x < y\}$

**Solution**:

Some elements in $\boldsymbol{r}$: $(0, 3), (-\pi, \pi), (-9, -3), (0.37, 8.43), (\sqrt{2}, \sqrt{3})$.

Dom $\boldsymbol{r} = \mathbb{R}$.
Reason: $\boldsymbol{r}$ is a relation on $\mathbb{R}$, and for every $x \in \mathbb{R}$ there exists $y$ such that $x < y$.

Im $\boldsymbol{r} = \mathbb{R}$.
Reason: For every $y \in \mathbb{R}$, $x < y$ for some $x \in \mathbb{R}$.

(8) $\boldsymbol{r}$ on $\mathbb{N}$ such that $a\boldsymbol{r}b$ if and only if $a \mid b$.

**Solution**:

Some elements in $\boldsymbol{r}$: $(2, 26), (6, 12), (3, 12), (15, 45), (475, 950)$.

Dom $\boldsymbol{r} = \mathbb{N}$.
Reason: $\boldsymbol{r}$ is a relation on $\mathbb{N}$, and every $a \in \mathbb{N}$ is the divisor of some $b \in \mathbb{N}$. (E.g., pick $b = 2a$).

Im $\boldsymbol{r} = \mathbb{N}$.
Reason: For every $b \in \mathbb{N}$ we can pick $a = 1$, which would satisfy the definition of "a divides b" given on pg. 43 of the textbook.

**Exercise 3.3**

(4) Determine whether the following $\boldsymbol{r}$ is reflexive, symmetric, transitive:

$\boldsymbol{r}$ on $\mathbb{Z}$ such that $a\boldsymbol{r}b$ if and only if $a \neq b$

**Solution**:

**Check reflexive**:

For $a\ \boldsymbol{r}\ a$ to hold, it requires $a \neq a$. Since this is impossible, $\boldsymbol{r}$ is not reflexive.

**Check symmetric**:

Let $m, n \in \mathbb{Z}$ such that $m\ \boldsymbol{r}\ n$.
Since $m \neq n$ implies $n \neq m$, it follows that $\boldsymbol{r}$ is symmetric.

**Check transitive**:

Let $m, n, p \in \mathbb{Z}$ such that $m\ \boldsymbol{r}\ n$ and $n\ \boldsymbol{r}\ p$.
This implies $m \neq n$ and $n \neq p$. BUT… can we conclude $m \neq p$?
NO. Here is a counterexample: let $a = 5, b = 2, c = 5$.
Then $a \neq b$ and $b \neq c$, but $a = c$. It follows that $\boldsymbol{r}$ is NOT transitive.

**Answers**: The given $\boldsymbol{r}$ is symmetric. But it is not reflexive and not transitive.