Homework due Sep. 22

Assigned exercises: 4.4: 2, 3, 4.
4.5: 2, 6, 7, 9, 12, 13, 17. (10 problems)
Graded exercises: 4.4: 2, 3. 4.5: 7, 9, 12.
Total possible points = 20.
3 pt each for 5 graded problems, plus 5 for completion of the rest.

Exercise 4.4

(2) Find the inverse of $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x + 10$. Show that $f^{-1} \circ f = f \circ f^{-1} = I_{\mathbb{Z}}$.

Solution:

Following standard algebraic steps for inverses, we get: $f^{-1}(x) = x - 10$.

For any $x \in \mathbb{Z}$ we have:

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = (x + 10) - 10 = x$$

and

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = (x - 10) + 10 = x$$

This shows $f^{-1} \circ f = f \circ f^{-1} = I_{\mathbb{Z}}$.

(3) Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} \frac{x-1}{2}, & \text{if } x \text{ is odd} \\ \frac{-x}{2}, & \text{if } x \text{ is even} \end{cases}$$

Find the inverse and show that it works.

Solution:

Plug in a few x-values and see what this function looks like:

x	1	2	3	4	5	6
$f(x)$	0	-1	1	-2	2	-3

Observation: When x is odd, $f(x) \geq 0$; when x is even, $f(x) < 0$.

Based on this observation, we conjecture the inverse $f^{-1} : \mathbb{Z} \to \mathbb{N}$ is defined by

$$f^{-1}(x) = \begin{cases} 2x + 1, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases}$$

To show that it works:

$$f^{-1}(f(x)) = \begin{cases} 2\left[\frac{x-1}{2}\right] + 1 = x, & \text{if } x \text{ is odd} \\ -2\left[\frac{-x}{2}\right] = x, & \text{if } x \text{ is even} \end{cases}$$

$$f(f^{-1}(x)) = \begin{cases} \frac{(2x+1)-1}{2} = x, & \text{if } x \geq 0 \\ \frac{-(-2x)}{2} = x, & \text{if } x < 0 \end{cases}$$

Exercise 4.5

(7) Let $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ be defined by $f(x, y) = x/y$. Determine whether f is (a) injective; (b) surjective.

Solution:

The given f is not injective.
For example, $f(1, 2) = f(2, 4) = 1/2$, but $(1, 2) \neq (2, 4)$.

f is surjective.

Reason: Any $w \in \mathbb{Q}$ is a rational number, and can be written as $w = x/y$ for some $x \in \mathbb{Z}$ and $y \in \mathbb{N}$, by the definition of rational number. Since $(x, y) \in \mathbb{Z}\times\mathbb{N}$ and $f(x, y) = w$, it follows that f is surjective.

(9) Prove: If $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f$ is injective.

Solution:

(1) Let A, B, C be sets, and f: $A \to B$ and g: $B \to C$ be injective functions.

(2) To show $g \circ f : A \to C$ is injective, suppose there exist $x_1, x_2 \in A$ such that $(g \circ f)(x_1) = (g \circ f)(x_2)$.

(3) Then $g(f(x_1)) = g(f(x_2))$.

(4) This implies $f(x_1) = f(x_2)$, since g is injective.

(5) Then we can conclude $x_1 = x_2$, because f is injective.

(6) From (2) and (5) it follows that $g \circ f$ is injective.

(12) Prove: Let $f : A \to B$ and $g : B \to C$ be functions. If $g \circ f$ is injective, then f is injective.

Solution:

(1) Let A, B, C be sets, and f: $A \to B$ and g: $B \to C$ be functions.

(2) Suppose $g \circ f$ is injective.

(3) To show $f : A \to B$ is injective, suppose there exist $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$.

(4) Then we can compose g on both sides and get: $g(f(x_1)) = g(f(x_2))$.

(5) This implies $x_1 = x_2$, since $g \circ f$ is injective.

(6) From (3) and (5) it follows that f is injective.