Homework due Sep. 15

Assigned exercises: 4.1: 2, 4, 6.
4.2: 2, 4, 6, 11, 12, 13, 16, 18, 19. (12 problems)
Graded exercises: 4.1: 4, 6. 4.2: 4, 11, 19.
Total possible points = 20.
3 pt each for 5 graded problems, plus 5 for completion of the rest.

Exercise 4.1

(4) List all functions from $\{1, 2\}$ to $\{1, 2, 3\}$.

Solution:

$f_1 = \{(1, 1), (2, 1)\}$

$f_2 = \{(1, 1), (2, 2)\}$

$f_3 = \{(1, 1), (2, 3)\}$

$f_4 = \{(1, 2), (2, 1)\}$

$f_5 = \{(1, 2), (2, 2)\}$

$f_6 = \{(1, 2), (2, 3)\}$

$f_7 = \{(1, 3), (2, 1)\}$

$f_8 = \{(1, 3), (2, 2)\}$

$f_9 = \{(1, 3), (2, 3)\}$

(6) Define the logical operator $\wedge : A^2 \to A$ as a function. (Here $A = \{T, F\}$.)

Solution:
The domain of $\wedge$ is $A^2 = \{(T, T), (T, F), (F, T), (F, F)\}$.
Since $\wedge$ outputs T only when the input is (T, T), the function values are:
$\wedge((T, T)) = T$
$\wedge((T, F)) = F$
$\wedge((F, T)) = F$
$\wedge((F, F)) = F$

Exercise 4.2

(4) $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x^2 - 1$, and $S = [-2, 2)$. Find the indicated images and/or pre-images.

Solution:

The graph of g is an upward opening parabola with minimum at $(0, -1)$. Its domain is $(-\infty, \infty)$ and range is $(-1, \infty)$.

(a) $g(S)$: On the interval $[-2, 2)$, the values of g range between -1 (when $x = 0$) and 3 (when $x = -2$). Therefore, $\boxed{g(S) = [-1, 3]}$

(b) $g^{-1}(S)$: The inverse image for y-values below -1 is empty, since they are outside the range. For y-values between -1 and 2 we get $-\sqrt{3} < x < \sqrt{3}$. Therefore, $\boxed{g^{-1}(S) = (-\sqrt{3}, \sqrt{3})}$

(c) $g(g^{-1}(S)) = g((-\sqrt{3}, \sqrt{3})) = [-1, 2)$, because $g(0) = -1$ and $g(\pm\sqrt{3}) = 2$. Answer: $\boxed{g(g^{-1}(S)) = [-1, 2)}$ Note that right end is open.

(d) $g^{-1}(g(S)) = g^{-1}([-1, 3]) = [-2, 2]$. Answer: $\boxed{g(g^{-1}(S)) = [-2, 2]}$

(11) Let $f : X \to Y$ and $A, B \subseteq X$. Prove: $f(A \cup B) = f(A) \cup f(B)$.

Solution:

(1) I will prove this by by showing subset inclusion both ways.

(2) Let X, Y be sets, $f : X \to Y$ be a function, and $A, B \subseteq X$.

(3) Proof of $f(A \cup B) \subseteq f(A) \cup f(B)$:

(3.1) Let $n \in f(A \cup B)$.

(3.2) Then there exisist m in $A \cup B$ such that $f(m) = n$. [defn. of image]

(3.3) This implies $m \in A$ or $m \in B$. [defn. of union]

(3.4) If $m \in A$ then $f(m) \in f(A)$, which is the same as saying $n \in f(A)$. This implies $n \in f(A) \cup f(B)$. [defn. of union]

(3.5) On the other hand, if $m \in B$ then $f(m) \in f(B)$, and so $n \in f(B)$. Again, this implies $n \in f(A) \cup f(B)$. [defn. of union]

(3.6) From (3.1), (3.4) and (3.5) we get: $n \in f(A \cup B) \Rightarrow n \in f(A) \cup f(B)$.

(3.7) It follows that $f(A \cup B) \subseteq f(A) \cup f(B)$.

(4) Proof of $f(A) \cup f(B) \subseteq f(A \cup B)$:

(4.1) Let $q \in f(A) \cup f(B)$.

(4.2) Then $q \in f(A)$ or $q \in f(B)$. [defn. of union]

(4.3) Consider each case: (i) $q \in f(A)$, (ii) $q \in f(B)$.

(i) If $q \in f(A)$: There is some $p \in A$ such that $f(p) = q$.
Since $p \in A$, we can say $p \in A \cup B$.
This means $f(p) \in f(A \cup B)$. That is: $q \in f(A \cup B)$.

(ii) If $q \in f(B)$: There is some $r \in B$ such that $f(r) = q$.
Since $r \in B$, we can say $r \in A \cup B$.
This means $f(r) \in f(A \cup B)$. That is: $q \in f(A \cup B)$

(4.4) It follows that if $q \in f(A) \cup f(B)$ then $q \in f(A \cup B)$. Thus $f(A) \cup f(B) \subseteq f(A \cup B)$.

(19) Let $f : X \to Y$ and $A \subseteq Y$. Disprove: $f(f^{-1}(A)) = A$.

Solution:

Here is a counterexample: Let $X = Y = \mathbb{R}$, $f(x) = x^2$, and let $A = [-1, 4]$. Then $f^{-1}(A) = [-2, 2]$, and $f(f^{-1}(A)) = [0, 4] \neq A$.