Homework due Sep. 10

Assigned exercises: 3.3: 8, 10, 12, 16, 18. 3.4: 4, 8, 10.
3.5: 2, 6, 7. (11 problems)
Graded exercises: 3.3: 10, 16. 3.4: 8. 3.5: 2, 7.
Total possible points = 20.
3 pt each for 5 graded problems, plus 5 for completion of the rest.

Exercise 3.3

(10) Given: $\boldsymbol{r}$ on $\mathbb{R}^2$ such that $(u, v)\boldsymbol{r}(x, y)$ iff $y - x^2 = v - u^2$. Determine whether $\boldsymbol{r}$ is reflexive, symmetric, transitive.

Solution:

(a) Check reflexive: Is $(u, v)\boldsymbol{r}(u, v)$ for all $(u, v) \in \mathbb{R}^2$?
Is $v - u^2 = v - u^2$ for all $(u, v) \in \mathbb{R}^2$? Yes, this is true.
Therefore, it is reflexive.

(b) Check symmetric:
Does $(u, v)\boldsymbol{r}(x, y) \;\Rightarrow\; (x, y)\boldsymbol{r}(u, v)$, for all $(x, y), (u, v) \in \mathbb{R}^2$?
Yes, since $y - x^2 = v - u^2 \;\Rightarrow\; v - u^2 = y - x^2$, for all $(x, y), (u, v) \in \mathbb{R}^2$.
Therefore, it is symmetric.

(c) Check transitive:
Does $(u, v)\boldsymbol{r}(x, y)$ and $(x, y)\boldsymbol{r}(p, q) \;\Rightarrow\; (u, v)\boldsymbol{r}(p, q)$, for all $(u, v), (x, y), (p, q) \in \mathbb{R}^2$?
Does $(y - x^2 = v - u^2$ and $q - p^2 = y - x^2) \;\Rightarrow\; (q - p^2 = v - u^2)$, for all $(u, v), (x, y), (p, q) \in \mathbb{R}^2$?
Yes, since this is always true, it is transitive.

Answers: The given $\boldsymbol{r}$ is reflexive, symmetric, and transitive.

(16) Prove: If $a \equiv b \bmod n$ and $c \equiv d \bmod n$, then $a + c \equiv b + d \bmod n$.

Solution:

Let $n \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$ such that $a \equiv b \bmod n$ and $c \equiv d \bmod n$.
By definition of mod: $a - b = k \cdot n$ and $c - d = l \cdot n$ for some $k, l \in \mathbb{Z}$.
Adding these two equations we get: $(a + c) - (b + d) = (k + l) \cdot n$.
This implies $a + c \equiv b + d \bmod n$ (by defn of mod), since $(k + l) \in \mathbb{Z}$.

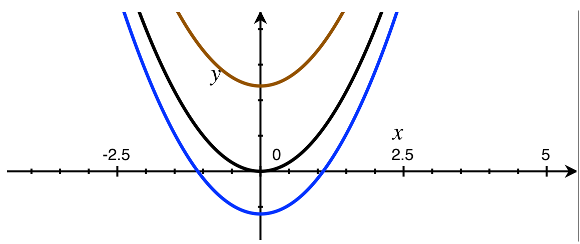
Exercise 3.4

(8) Given: $\boldsymbol{r}$ on $\mathbb{R}^2$ such that $(u, v)\boldsymbol{r}(x, y)$ iff $y - x^2 = v - u^2$.
Find all the equivalence classes.

Solution:

Equivalence classes typically exhibit some sort of pattern. Let's try some explorations and see if we can find one.

Suppose $(u, v) = (0, 0)$:
Then every related (x, y) must satisfy $y - x^2 = 0$, or $y = x^2$.
Thus the equivalence class of $(0, 0)$ is: $E_{(0,0)} = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$
Suppose $(u, v) = (1, 2)$:
Then every related (x, y) must satisfy $y - x^2 = 2 - 1^2$, or $y = x^2 + 1$.
The equivalence class of $(1, 2)$ is: $E_{(1,2)} = \{(x, y) \in \mathbb{R}^2 \mid y = x^2 + 1\}$
Suppose $(u, v) = (-3, 3)$:
Then every related (x, y) must satisfy $y - x^2 = 3 - (-3)^2$, or $y = x^2 - 6$.
The equivalence class of $(-3, 3)$ is: $E_{(-3,3)} = \{(x, y) \in \mathbb{R}^2 \mid y = x^2 - 6\}$
Thus, we can conclude the equivalence classes of $\boldsymbol{r}$ are parabolas of the form $y = x^2 + c$ for $c \in \mathbb{R}$. Some samples are shown in the graph below:



Exercise 3.3

(2) Let $\mathcal{A} = \{A_\gamma\}_{\gamma\in\Gamma}$ be an indexed collection of sets. Prove: $\left(\bigcap_{\gamma\in\Gamma} A_\gamma\right)^C = \bigcup_{\gamma\in\Gamma} A_\gamma^C$.

Solution:
(1) I will prove this by by showing subset inclusion both ways.
(2) Let $\mathcal{A} = \{A_\gamma\}_{\gamma\in\Gamma}$ be an indexed collection of sets.
(3) Proof of $\left(\bigcap_{\gamma\in\Gamma} A_\gamma\right)^C \subseteq \bigcup_{\gamma\in\Gamma} A_\gamma^C$:
(3.1) Let $x \in \left(\bigcap_{\gamma\in\Gamma} A_\gamma\right)^C$.
(3.2) Then $x \notin \bigcap_{\gamma\in\Gamma} A_\gamma$. [definition of complement]
(3.3) This implies $x \notin A_\gamma$ for some $\gamma \in \Gamma$. [negation of intersection defn.]
(3.4) In other words, $x \in A_\gamma^C$ for some $\gamma \in \Gamma$. [defn. of complement]
(3.5) This implies $x \in \bigcup_{\gamma\in\Gamma} A_\gamma^C$. [defn. of union]
(3.6) From (3.1) and (3.5) we get: $x \in \left(\bigcap_{\gamma\in\Gamma} A_\gamma\right)^C \Rightarrow x \in \bigcup_{\gamma\in\Gamma} A_\gamma^C$.
(3.7) It follows that $\left(\bigcap_{\gamma\in\Gamma} A_\gamma\right)^C \subseteq \bigcup_{\gamma\in\Gamma} A_\gamma^C$ [defn. of subset]

(4) Proof of $\bigcup_{\gamma\in\Gamma} A_\gamma^C \subseteq \left(\bigcap_{\gamma\in\Gamma} A_\gamma\right)^C$:
(4.1) Let $s \in \bigcup_{\gamma\in\Gamma} A_\gamma^C$.
(4.2) Then, for some $\gamma \in \Gamma$: $s \in A_\gamma^C$. [defn. of union]
(4.3) For some $\gamma \in \Gamma$: $s \notin A_\gamma$. [defn. of complement]
(4.4) For some $\gamma \in \Gamma$, $s \notin A_\gamma$ implies $s \notin \bigcap_{\gamma\in\Gamma} A_\gamma$.

[negation of intersection defn.]

(4.5) Thus $s \in \left(\bigcap_{\gamma\in\Gamma} A_\gamma\right)^C$. [defn. of complement]

(4.6) Lines (4.1) and (4.6) show that $s \in \bigcup_{\gamma\in\Gamma} A_\gamma^C \Rightarrow s \in \left(\bigcap_{\gamma\in\Gamma} A_\gamma\right)^C$.

Therefore, $\bigcup_{\gamma\in\Gamma} A_\gamma^C \subseteq \left(\bigcap_{\gamma\in\Gamma} A_\gamma\right)^C$ [defn. of subset]

(5) It follows from (3) and (4) that $\left(\bigcap_{\gamma\in\Gamma} A_\gamma\right)^C = \bigcup_{\gamma\in\Gamma} A_\gamma^C$.

[definition of set equality]

(7) Find $\displaystyle\bigcap_{k\in\mathbb{N}} \{m \in \mathbb{N} \mid m \leq k\}$.

Solution:

Let's write out and see what the first few sets in this collection look like.

$k = 1$: $\{m \in \mathbb{N} \mid m \leq 1\} = \{1\}$

$k = 2$: $\{m \in \mathbb{N} \mid m \leq 2\} = \{1, 2\}$

$k = 3$: $\{m \in \mathbb{N} \mid m \leq 3\} = \{1, 2, 3\}$

As k increases, the sets continue to include more and more elements of $\mathbb{N}$.

The only element that every set contains in common is 1. Therefore,

$$\bigcap_{k\in\mathbb{N}} \{m \in \mathbb{N} \mid m \leq k\} = \{1\}$$