Homework due date Aug. 30

Assigned exercises: 2.7: 1. 2.8: 6, 7, 10, 12.
2.9: 6, 11, 13, 14, 20. (10 problems)
Graded exercises: 2.7: 1. 2.8: 6, 12. 2.9: 6, 11.
Total (maximum) possible points = 20.
3 pt for each of 5 graded problems, plus 5 for completion of the rest.

Exercise 2.7

(1) Prove: $A - B = \emptyset$ iff $A \subseteq B$.

Solution:

(a) Let A and B be sets.

(b) Since this is a 2-way implication, it must be proved in both directions.

(c) Proof of $A - B = \emptyset \Rightarrow A \subseteq B$:

i. Let $A - B = \emptyset$.
ii. To prove $A \subseteq B$, suppose $x \in A$.
iii. Then, $x \notin A - B$. [since $A - B$ is empty]
iv. This implies $x \notin A$ or $x \in B$. [by negating defn of $x \in A - B$]
v. From line ii. we know $x \in A$. Thus, line iv. implies $x \in B$.
vi. From lines ii. and v. we have $x \in A \Rightarrow x \in B$, and it follows that $A \subseteq B$.

(d) Proof of $A \subseteq B \Rightarrow A - B = \emptyset$:

i. I will prove the contrapositive: $A - B \neq \emptyset \Rightarrow A \nsubseteq B$
ii. Suppose $A - B \neq \emptyset$ and let $y \in A - B$.
iii. This implies $y \in A$ and $y \notin B$. [by defn of $y \in A - B$]
iv. Then $A \nsubseteq B$, because A contains an element that is not in B.
v. From lines i. and iv. we have: $A - B \neq \emptyset \Rightarrow A \nsubseteq B$.
vi. It follows that $A \subseteq B \Rightarrow A - B = \emptyset$.

(e) From (c) and (d) we get: $A - B = \emptyset$ iff $A \subseteq B$.

Exercise 2.8

(6) If $C \cap (A \cap B) = \emptyset$ then $C \cap A = \emptyset$ or $C \cap B = \emptyset$.

Solution:

This is false.
Here is a counterexample: Let $A = \{1, 2, 3\}, B = \{2, 4\}, C = \{3, 4\}$.
Then $C \cap (A \cap B) = \{3, 4\} \cap \{2\} = \emptyset$.
But $C \cap A = \{3\}$ and $C \cap B = \{4\}$, neither of which is empty.

(12) If $A \subseteq B$ then $A \cup C \subseteq B \cup C$.

Solution:

This is true. Here is a proof:

Let A, B, C be sets such that $A \subseteq B$.
Suppose $m \in A \cup C$. Then $m \in A$ or $m \in C$, by defn of subsets. Consider each case separately.

Suppose $m \in A$: Then $m \in B$, since $A \subseteq B$. In turn, that means $m \in B \cup C$, which shows $m \in A \cup C \Rightarrow m \in B \cup C$. Thus $A \cup C \subseteq B \cup C$.

On the other hand, suppose $m \in C$: Then it immediately follows that $m \in C \cup B$, which means $m \in A \cup C \Rightarrow m \in B \cup C$. Thus $A \cup C \subseteq B \cup C$.

This proves that if $A \subseteq B$ then $A \cup C \subseteq B \cup C$.

Exercise 2.9

(6) Prove: For each set A, $A \cap \emptyset = \emptyset$.

Solution:

(a) Let A be a set.

(b) To prove $A \cap \emptyset = \emptyset$, we must show subset both ways.

(c) Proof of $A \cap \emptyset \subseteq \emptyset$:

   i. By way of contradiction, suppose $A \cap \emptyset \nsubseteq \emptyset$.
   ii. Then there exists some $x \in A \cap \emptyset$.
   iii. Then, by definition, $x \in A$ and $x \in \emptyset$.
   iv. But that is a contradiction, since x cannot be in the empty set.
   v. It follows that $A \cap \emptyset \subseteq \emptyset$.

(d) Proof of $\emptyset \subseteq A \cap \emptyset$:

   i. By Theorem 2.1, page 24, $\emptyset \subseteq$ of every set. Therefore, $\emptyset \subseteq A \cap \emptyset$.

(e) From lines (c), (d), we have: $A \cap \emptyset = \emptyset$.

(11) Prove: $A \subseteq B$ iff $A \cap B = A$.

Solution:

(1) I will prove this directly. Since it is an "if and only if" claim, the implication must be proved in both directions.

(2) Proof of $A \subseteq B \Rightarrow A \cap B = A$.

(2.1) Let $A \subseteq B$.

(2.2) To prove $A \cap B = A$, we must show subset both ways.

(2.3) Proof of $A \cap B \subseteq A$:

Let $m \in A \cap B$.
Then $m \in A$ and $m \in B$ (intersection definition), which means $m \in A \cap B \Rightarrow m \in A$. So $A \cap B \subseteq A$.

(2.4) Proof of $A \subseteq A \cap B$:

Suppose $y \in A$. Then $y \in B$, since $A \subseteq B$ by line (2.1).

Thus $y \in A$ and $y \in B$, which means $y \in A \cap B$.
It follows that $y \in A \Rightarrow y \in A \cap B$, so that $A \subseteq A \cap B$.

(2.5) From lines (2.3)-(2.4) we conclude $A \cap B = A$,
and it follows that $A \subseteq B \Rightarrow A \cap B = A$.

(3) Proof of converse: $A \cap B = A \Rightarrow A \subseteq B$:

(3.1) Let A and B be sets such that $A \cap B = A$.
(3.2) To prove $A \subseteq B$, let $x \in A$.
(3.3) Then $x \in A \cap B$, since $A = A \cap B$ by hypothesis.
(3.4) This implies $x \in B$. [defn of intersection]
(3.5) Thus $x \in A \Rightarrow x \in B$, and we have $A \subseteq B$.
(3.6) From lines (3.1) and (3.5), $A \cap B = A \Rightarrow A \subseteq B$.

(4) The proofs in (2) and (3) show that $A \subseteq B$ if and only if $A \cap B = A$.