Student name:

MATH 288: Intro to Proof — **Final exam**
Fall 2021 — **September. 27, 2021**

Instructions:

- Answer all questions on separate paper (not on this sheet!).
- This is a regular "closed-book" test, and is to be taken without the use of notes, books, electronic devices, or other reference materials.
- This test contains questions numbered (1) to (10). It adds up to 50 points.

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(1) [4 pts.] Determine whether the following function is one-to-one, onto, neither, or both

$$f : \mathbb{N} \to (0, 1] \text{ defined by } f(n) = 1/n$$

Be sure to include formal justification for all your claims.

(2) [4 pts.] Using mathematical induction prove that $\displaystyle\sum_{k=1}^{n} k\, 2^{k-1} = (n-1)2^n + 1$ for all $n \in \mathbb{N}$. Show all steps in the formal process of an induction proof.

(3) [2 pts. each$\times 3 = 6$ pts.] Negate the following statements. Negation must be written in complete words and sentences, with sparing use of symbolism. (Yes, you may use the standard arithmetic symbols used in the questions themselves –, =, ≤, etc.)

(a) For each real number r there is a real number s such that $rs = 1$.

(b) For all $r, s \in \mathbb{R}$, if $r < 0$ and $s < 0$ then $r + s < 0$ and $rs > 0$.

(c) If l is a line and P is a point not on it, then there is exactly one line passing through P that is parallel to l.

(4) [3 pts. each$\times 2 = 6$ pts.] Prove or disprove each of the following assertions

(a) There exists a subset S of $\mathbb{N}$ such that $n \in S$ implies $n + 1 \in S$, but $S \neq \mathbb{N}$. [Hint: If your solution requires more than 3-4 carefully crafted lines, there is high probability it is wrong!]

(b) Let A and B be sets, and let $f : A \to B$ be a function. If $C \subseteq A$, then $f(A - C) \subseteq f(A) - f(C)$.

(5) [2 pts. each$\times 3 = 6$ pts.] Let $A = \{4, 5, 6\}$ and $B = \{p, q, r\}$. Define the functions $f = \{(4, p), (5, q), (6, p)\}$ and $g = \{(p, 4), (q, 6), (r, 5)\}$. Describe each of the following functions by listing its ordered pairs, and state its domain and codomain.

(a) g^{-1} (b) $f \circ g$ (c) $f \circ g \circ f$

(6) [3 pts. each$\times 2 = 6$ pts.] For each relation given below, determine whether it is reflexive, symmetric, and transitive. I hope it is obvious that a meaningful solution will require much more than just a "yes" or "no" answer!

(a) The relation s on $\mathbb{R}^2$ such that $(u, v)\,\mathsf{s}\,(x, y)$ if and only if $uv \neq xy$.

(b) The relation $r = \{(1,1),(2,2),(3,3),(4,4),(2,3),(3,2)\}$ on the set $A = \{1, 2, 3, 4\}$.

Attempt any 3 of the remaining questions. This is NOT an opportunity for earning extra credit – if you do all 4, I'll grade the first 3 that you turn in!

(7) [6 pts.] Let S be a set and $\sim$ an equivalence relation on it. For any $a, b \in S$ show that $a \in E_b$ if and only if $E_a = E_b$ (where E_a and E_b denote equivalence classes of a and b).

(8) [6 pts.] Suppose $f : A \to B$ and $g : B \to C$ are functions. Each of the following statements is either true or false. If it is true prove it. If it is not true, indicate what further hypotheses are needed to make it true, and prove your claim.

(a) If $g \circ f$ is one-to-one, then f must be one-to-one.

(b) If $g \circ f$ is one-to-one, then g must be one-to-one.

(9) [6 pts.] Let A, B and C be sets. Prove that if $(A \cup B) \subseteq C$ and $(C \setminus B) \subseteq (C \setminus A)$, then $A \subseteq B$.

(10) [6 pts.] Let A and B be sets and let $f : A \to B$ be a function. Suppose S and T are subsets of A. Prove or disprove each of the following
(a) If $S \subseteq T$, then $f(S) \subseteq f(T)$.
(b) If $f(S) \subseteq f(T)$, then $S \subseteq T$.

End of test

Intro to Proof: Fall 2021: Final exam solutions

(1) [4 pts.] Determine whether the following function is one-to-one, onto, neither, or both

$$f : \mathbb{N} \to (0, 1] \text{ defined by } f(n) = 1/n$$

Be sure to include formal justification for all your claims.

Solution: I conjecture that f is one-to-one, but it is not onto.

Verification of one-to-one: Suppose $f(n_1) = f(n_2)$ for some $n_1, n_2 \in \mathbb{N}$.
This implies $1/n_1 = 1/n_2$. Thus $n_1 = n_2$, and it follows that f is one-to-one.

Counterexample to show f is NOT onto:
Let $y = 3/4$. Then $y \in (0, 1]$, the codomain of f.
But there is no $n \in \mathbb{N}$ such that $1/n = 3/4$. It follows that f is not onto.

Grade: 1+1 point for correct answer to one-to-one + onto.
1+1 point for correct justification of each claim.

(2) [4 pts.] Using mathematical induction prove that $\displaystyle\sum_{k=1}^{n} k\ 2^{k-1} = (n-1)2^n + 1$

for all $n \in \mathbb{N}$. Show all steps in the formal process of an induction proof.

Solution:

Let $T = \{n \in \mathbb{N} \mid \sum_{k=1}^{n} k\ 2^{k-1} = (n-1)2^n + 1\}$.
Basis for induction: Let $n = 1$.
Then, the left side is: $1 \cdot 2^0 = 1$.
The right side is: $(1-1)2^1 + 1 = 0 \cdot 2 + 1 = 1$.
Thus, $1 \in T$, since $\sum_{k=1}^{n} k\ 2^{k-1} = (n-1)2^n + 1$ holds for $n = 1$.
Induction step: Let $n \in T$ for some $n \in \mathbb{N}$.
Then, for that n we have: $\sum_{k=1}^{n} k\ 2^{k-1} = (n-1)2^n + 1$
The $(n+1)$th term is $(n+1)2^n$, which we add to both sides and get

$$\begin{aligned}\textstyle\sum_{k=1}^{n} k\ 2^{k-1} + (n+1)2^n &= (n-1)2^n + 1 + (n+1)2^n \\ &= 2^n(n-1\ +\ n+1) + 1 \\ &= 2^n(2n) + 1 \\ &= n\ 2^{n+1} + 1\end{aligned}$$

This shows $n \in T \;\Rightarrow\; (n+1) \in T$

Since $1 \in T$, and $n \in T \;\Rightarrow\; (n+1) \in T$, by the principle of mathematical induction, $T = \mathbb{N}$. Therefore, $\displaystyle\sum_{k=1}^{n} k\ 2^{k-1} = (n-1)2^n + 1$ for all $n \in \mathbb{N}$.

Grade: 1 pt = show/implement correct base step with $n = 1$.
1 pt = state correct induction hypothesis.
1.5 pt = carry out induction step and show $n \in T \;\Rightarrow\; (n+1) \in T$.
0.5 pt = close the argument by correcly invoking principle of induction.

(3) [2 pts. each×3 = 6 pts.] Negate the following statements:

(a) For each real number r there is a real number s such that $rs = 1$.
Solution: There exists a real number r such that for all real numbers s, $rs \neq 1$.

(b) For all $r, s \in \mathbb{R}$, if $r < 0$ and $s < 0$ then $r + s < 0$ and $rs > 0$.
Solution: There exist $r, s \in \mathbb{R}$ such that $r < 0$ and $s < 0$, for which $r + s \geq 0$ or $rs \leq 0$.

(c) If l is a line and P is a point not on it, then there is exactly one line passing through P that is parallel to l.
Solution: There exists a line l and a point P not on it, such that more than one line parallel to l passes through P (or, there is no parallel line to l that passes through P).

Grade: No partial credit, unless logical structure is mostly correct.

(4) [3 pts. each×2 = 6 pts.] Prove or disprove each of the following assertions

(a) There exists a subset S of $\mathbb{N}$ such that $n \in S$ implies $n + 1 \in S$, but $S \neq \mathbb{N}$.
Solution: The assertion is true. Here is a proof:
Let $S = \mathbb{N} - \{1, 2\} = \{3, 4, 5, \cdots\}$.
Then for every $n \in S$ it is true that $n + 1 \in S$.
But $\mathbb{N} \nsubseteq S$, because $1 \in \mathbb{N}$ but $1 \notin S$. Therefore, $S \neq \mathbb{N}$.

(b) Let A and B be sets, and let $f : A \to B$ be a function. If $C \subseteq A$, then $f(A - C) \subseteq f(A) - f(C)$.
Solution: The assertion is false. Here is a proof/counterexample:
Let $A\{1, 2\}$, $B = \{p\}$, $f = \{(1, p), (2, p)\}$, $C = \{2\}$.
Then $f : A \to B$ is a function and $C \subseteq A$.
$f(A - C) = f(\{1\}) = \{p\}$. And $f(A) = f(C) = \{p\}$ implies $f(A) - f(C) = \emptyset$.
Therefore, $f(A - C) \nsubseteq f(A) - f(C)$, because $\{p\} \nsubseteq \emptyset$.

Grade: 3 points each. General yardstick: 1.5 pt for correct claim of true/false; 1.5 pt for correct proof/counterexample.

(5) [2 pts. each×3 = 6 pts.] Let $A = \{4, 5, 6\}$ and $B = \{p, q, r\}$. Define the functions $f = \{(4, p), (5, q), (6, p)\}$ and $g = \{(p, 4), (q, 6), (r, 5)\}$. Describe each of the following functions by listing its ordered pairs, and state its domain and codomain.

(a) g^{-1} (b) $f \circ g$ (c) $f \circ g \circ f$

Solution:

(a) $\boxed{g^{-1} = \{(4, p), (6, q), (5, r)\}.}$
Its domain is A and codomain is B, since it is the inverse of $g : B \to A$.

(b) Since $g : B \to A$ and $f : A \to B$, we have $f \circ g : B \to B$.
In other words, its domain and codomain are both B.
Since $(f \circ g)(x) = f(g(x))$, we get: $\boxed{f \circ g = \{(p, p), (q, p), (r, q)\}.}$

(c) $f \circ g \circ f$ has the domain of f, since it is the innermost function, and the codomain of f as well, since it is the outermost function. Thus, we have $f \circ g \circ f : A \to B$.
Since $(f \circ g \circ f)(x) = f(g(f(x)))$, we get: $\boxed{f \circ g \circ f = \{(4, p), (5, p), (6, p)\}.}$

Grade: 2 points each. General yardstick: 1 pt for correct domain and codomain; 1 pt for correct set of ordered pairs in the answer.

(6) [3 pts. each×2 = 6 pts.] For each relation given below, determine whether it is reflexive, symmetric, and transitive. I hope it is obvious that a meaningful solution will require much more than just a "yes" or "no" answer!

(a) The relation $\mathtt{s}$ on $\mathbb{R}^2$ such that $(u, v)\,\mathtt{s}\,(x, y)$ if and only if $uv \neq xy$.

Solution:

Reflexive: Let $(u, v) \in \mathbb{R}^2$. Then $(u, v)\mathtt{s}(u, v)$ would imply $uv \neq uv$, which is impossible. Thus, $\mathtt{s}$ is NOT reflexive.

Symmetric: Let $(u, v), (x, y) \in \mathbb{R}^2$ such that $(u, v)\mathtt{s}(x, y)$.
Then, $uv \neq xy$ by definition of $\mathtt{s}$.
That means $xy \neq uv$, and we can conclude $(x, y)\mathtt{s}(u, v)$. So it is symmetric.

Transitive: Let $(u, v), (x, y), (p, q) \in \mathbb{R}^2$ such that $(u, v)\mathtt{s}(x, y)$ and $(x, y)\mathtt{s}(p, q)$.
Then we have $uv \neq xy$ and $xy \neq pq$. But this does not necessarily mean $uv \neq pq$.
[e.g., $(u, v) = (1, 2), (x, y) = (3, 4), (p, q) = (2, 1)$]
So $\mathtt{s}$ is NOT transitive.

$\boxed{\text{It follows that } \mathtt{s} \text{ is symmetric, but neither reflexive nor transitive.}}$

(b) The relation $r = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 2)\}$ on the set $A = \{1, 2, 3, 4\}$.

Solution:

Reflexive: For every $m \in A$, we see that $(m, m) \in r$. So it is reflexive.

Symmetric: We also see that for any $m, n \in A$, if $(m, n) \in r$ then $(n, m) \in r$.
[E.g., $(2, 3) \in r \Rightarrow (3, 2) \in r$.] So it is symmetric.

Transitive: Similarly, $(m, n) \in r$ and $(n, p) \in r \Rightarrow (m, p) \in r$ for every $m, n, p \in A$. So it is transitive.

$\boxed{\text{It follows that } r \text{ is reflexive, symmetric, and transitive on } A.}$

Grade: 3 points each. General yardstick: 1 pt each for correct analysis of reflexive, symmetric, transitive. 50/50 split between answer + reasoning.

(7) [6 pts.] Let S be a set and $\sim$ an equivalence relation on it. For any $a, b \in S$ show that $a \in E_b$ if and only if $E_a = E_b$ (where E_a and E_b denote equivalence classes of a and b).

Solution:

Let S be a set and $\sim$ an equivalence relation on it.
Let $a, b \in S$, and let E_a and E_b respectively denote their equivalence classes.
There are two implications that must be proved here:

(i) If $a \in E_b$ then $E_a = E_b$.

(ii) If $E_a = E_b$ then $a \in E_b$.

Proof of (i): If $a \in E_b$ then $E_a = E_b$.

Let $a \in E_b$.
Now, to prove $E_a = E_b$, let $x \in E_a$.
Then $a \sim x$, by defn of E_a. We also note that $b \sim a$, since $a \in E_b$.
Putting together we get: $b \sim x$, since $b \sim a$ and $a \sim x$, and $\sim$ is transitive.
Since $b \sim x$ we conclude $x \in E_b$, from which it follows that $E_a \subseteq E_b$.

To prove $E_b \subseteq E_a$, suppose $y \in E_b$.
Then $b \sim y$ by defn of E_b.
We also know $b \sim a$, since $a \in E_b$. And $a \sim b$ due to symmetry.
Thus we get: $a \sim y$, since $a \sim b$ and $b \sim y$, and $\sim$ is transitive.
This means $y \in E_a$, and it follows that $E_b \subseteq E_a$.

Since $E_a \subseteq E_b$ and $E_b \subseteq E_a$, we have $E_a = E_b$.

This completes the proof of (i).

Proof of (ii): If $E_a = E_b$ then $a \in E_b$.
Suppose $E_a = E_b$.
We know that $a \in E_a$, since $\sim$ is reflexive and a is related to itself.
Since $E_a = E_b$ it follows that $a \in E_b$.

This completes the proof of (ii).

Grade: 1 pt = attempt to prove correct 2 implications.
1 pt = correctly state initial hypotheses (about S, $\sim$, $a, b \in S$, etc.)
1 pt = correctly prove implication (ii).
3 pt = correctly prove implication (i).

(8) [6 pts.] Suppose $f : A \to B$ and $g : B \to C$ are functions. Each of the following statements is either true or false. If it is true prove it. If it is not true, indicate what further hypotheses are needed to make it true, and prove your claim.

(a) If $g \circ f$ is one-to-one, then f must be one-to-one.

(b) If $g \circ f$ is one-to-one, then g must be one-to-one.

Solution:

(a) This statement is true. Here is a proof:
Let A, B, C be sets, and f: $A \to B$ and g: $B \to C$ be functions.
Let $g \circ f : A \to C$ be one-to-one.
Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$.
Then $g(f(x_1)) = g(f(x_2))$, since $f(x_1), f(x_2)$ is in the domain of g.
This implies $x_1 = x_2$, since $g \circ f$ is one-to-one.
Thus $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, and it follows that f is one-to-one.

(b) This statement is false. Here is a counterexample:
Let $f : \mathbb{R}^+ \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$ and $g(x) = x^2$.
Then $g \circ f : \mathbb{R}^+ \to \mathbb{R}$ is one-to-one, since $(g \circ f)(x) = x$.
But g is not one-to-one.
An additional hypothesis that would make the statement true is: If f is onto.
New statement: If $g \circ f$ is one-to-one and f is onto, then g must be one-to-one.
Proof: Let A, B, C be sets, and f: $A \to B$ and g: $B \to C$ be functions.
Let $g \circ f : A \to C$ be one-to-one, and f be onto.
Suppose $g(y_1) = g(y_2)$ for some $y_1, y_2 \in B$.
Since f is onto, there exist $x_1, x_2 \in A$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.
Then $g(f(x_1)) = g(f(x_2))$, since $g(y_1) = g(y_2)$.
This implies $x_1 = x_2$, since $g \circ f$ is one-to-one.
If $x_1 = x_2$ it follows that $f(x_1) = f(x_2)$, since f is a function.

This implies $y_1 = y_2$, since $f(x_1) = y_1$ and $f(x_2) = y_2$.
Thus $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$, and it follows that g is one-to-one.

Grade: (a) = 2 points, and (b) = 4 points.
For (b): 1 pt = say the statement is false.
1 pt = provide counterexample and show it is valid.
1 pt = state additional hypothesis that makes statement true.
1 pt = prove resulting implication correctly.

(9) [6 pts.] Let A, B and C be sets. Prove that if $(A \cup B) \subseteq C$ and $(C \setminus B) \subseteq (C \setminus A)$, then $A \subseteq B$.

Solution:
Suppose A, B and C are sets that satisfy: $(A \cup B) \subseteq C$ and $(C \setminus B) \subseteq (C \setminus A)$.

To show $A \subseteq B$, consider any $x \in A$.
Then $x \in (A \cup B)$, by definition of union.
This implies $x \in C$, since $(A \cup B) \subseteq C$.
Thus, we can conclude $x \notin (C \setminus A)$, because we know $x \in C$ and $x \in A$.
This would imply $x \notin (C \setminus B)$, since $(C \setminus B) \subseteq (C \setminus A)$.
Negation of $x \in (C \setminus B)$ implies $x \notin C$ or $x \in B$.
But we have already showed $x \in C$. So it follows that $x \in B$.
Therefore, we have shown $x \in A \Rightarrow x \in B$.
By definition, then $A \subseteq B$.

Grade: 3 pt = correct, complete hypotheses, and start of element argument.
3 pt = correctly complete the argument, and provide closure.

(10) [6 pts.] Let A and B be sets and let $f : A \to B$ be a function. Suppose S and T are subsets of A. Prove or disprove each of the following
(a) If $S \subseteq T$, then $f(S) \subseteq f(T)$.
(b) If $f(S) \subseteq f(T)$, then $S \subseteq T$.

Solution:
(a) This statement is true. Here is a proof (direct):
(1) Let A and B be sets and let $f : A \to B$ be a function.
(2) Let $S, T \subseteq A$ such that $S \subseteq T$.
(3) I'll show that $f(S) \subseteq f(T)$.
(4) Consider any $n \in f(S)$.
(5) There exists $m \in S$ such that $f(m) = n$. [by definition of image of S]
(6) $m \in S$ implies $m \in T$. [since $S \subseteq T$ by hypothesis]
(7) Since $m \in T$, it follows that $f(m) \in f(T)$ by definition of image of T.
(8) Thus $n \in f(T)$, since $f(m) = n$.
(9) From lines (4) and (8) it follows that $f(S) \subseteq f(T)$.

(b) This is false. Here is a proof (counterexample):
Let $A = B = \mathbb{Z}$ and let $f : A \to B$ be given by $f(x) = x^2$.
Consider $S = \{1, 2\}$ and $T = \{-2, -1\}$
Then S and T are subsets of A, and $f(S) = f(T) = \{1, 4\}$.
Therefore, it is true that $f(S) \subseteq f(T)$.
But $S \not\subseteq T$.

Grade: (a) = 4 points, and (b) = 2 points.