A warmup

This warmup may be a bit of a challenge, but let's try it and see!

Let $\boldsymbol{r}$ be a relation on $\mathbb{Z} \times \mathbb{N}$ defined by $(x, y)\boldsymbol{r}(u, v)$ iff $xv = yu$.

1. List a few sample elements in the domain of $\boldsymbol{r}$.
2. What are the domain and image set of $\boldsymbol{r}$?
3. Do a quick check/proof to convince yourself that $\boldsymbol{r}$ is an equivalence relation.
4. Find the equivalence class of each of the following:
   (a) $(1, 2)$
   (b) $(2, 1)$
   (c) $(-5, 2)$
   (d) $(0, 3)$

A general hint: It might be easier to do these tasks if you rewrite the defining condition of $\boldsymbol{r}$ in a ratio/fraction form.

Moral of the story:

We can formally define the set of rational numbers $\mathbb{Q}$ as equivalence classes of the above $\boldsymbol{r}$ on $\mathbb{Z} \times \mathbb{N}$.

Some explorations

1. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 4, 5, 6\}$, $C = \{3, 4, 5, 6, 7\}$, $D = \{4, 5, 6, 7, 8\}$.

   (a) Find $A \cup B \cup C \cup D$.

   (b) Find $A \cap B \cap C \cap D$.

2. For each $n \in \mathbb{N}$, consider the set $C_n = \{n, n+1, n+2, n+3, n+4\}$. Next, let's define the "family" (or "collection") of sets

   $$C = \{C_1, C_2, C_3, C_4\}$$

   (a) Find $\bigcup_{k=1}^{4} C_k$.

   (b) Find $\bigcap_{k=1}^{4} C_k$.

   (c) Now consider the set $K = \{1, 2, 3, 4\}$, and carefully examine the following definitions

   $$\bigcup_{k=1}^{4} C_k = \{x \mid x \in C_k \text{ for some } k \in K\}$$

   $$\bigcap_{k=1}^{4} C_k = \{x \mid x \in C_k \text{ for all } k \in K\}$$

   How does your solution process in parts (a) and (b) compare with these definitions?

3. Exercise: For each $n \in \mathbb{N}$, let $A_n = \{1, n, n^2\}$. For example, $A_1 = \{1\}$, $A_2 = \{1, 2, 4\}$, $A_3 = \{1, 3, 9\}$, etc.

   (a) Find $\bigcup_{k=1}^{4} A_k$.

   (b) Find $\bigcap_{k=1}^{4} A_k$.

   (c) Find $\bigcup_{k=1}^{\infty} A_k$.

   (d) Find $\bigcap_{k=1}^{\infty} A_k$.