Induction worksheet

For each exercise that uses summation notation, rewrite the sum in expanded form, and then prove the indicated result.

1. Prove that $\sum_{k=1}^{n} 2^{k-1} = 2^n - 1$ for all $n \in \mathbb{N}$.
2. Prove that $\sum_{k=1}^{n} (3k-1) = \frac{1}{2}n(3n+1)$ for all $n \in \mathbb{N}$.
3. Prove that $\sum_{k=1}^{n} \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ for all $n \in \mathbb{N}$.
4. Prove that 8 divides $5^{2n} - 1$ for all $n \in \mathbb{N}$.
5. Prove that 5 divides $9^n - 4^n$ for all $n \in \mathbb{N}$.
6. The following inequality holds for all $n \in \mathbb{N}$ larger than some k:

   $$n^2 \leq 2^n$$

   Find k and prove the inequality using induction.
7. What justifies the use of induction in the above question, where the base case is not 1?

Just FYI

There is a closely related theorem known as the "principle of strong induction." It states:

Let T be a subset of $\mathbb{N}$ such that

(a) $1 \in T$; and

(b) If $1, 2, \ldots, n \in T$, then $(n+1) \in T$ (for all $n \in \mathbb{N}$)

Then $T = \mathbb{N}$.

Practice proving a useful theorem

Theorem (informal statement): Any pair of equivalence classes is either identical or disjoint.

Proof strategy:

1. The theorem has the form: $p \Rightarrow q$ or r
   It is helpful to first identify p, q and r. What are they?

2. One way to prove this is to suppose p is true and q is not true. Then we must show r is true.

3. Formally:

   (a) Let $\sim$ be an equivalence relation on set A, and let $u, v \in A$.

   (b) Let E_u, E_v respectively denote the equivalence classes of u, v.

   (c) Suppose $E_u \cap E_v \neq \emptyset$. We will then show ____________________

   (d) This requires showing: ____ $\subseteq$ ____, and ____ $\supseteq$ ____,

   (e) Proof of $\subseteq$:

      i. Let $x \in$ ____
      ii. Then $u \sim x$ (u is related to x) (why? ____________________)
      iii. This means $x \sim u$ (why? ____________________)
      iv. Since $E_u \cap E_v \neq \emptyset$, there exists some $m \in E_u \cap E_v$, and we can show $x \sim m$ (how? ____________________)
      v. We also know $m \sim v$ (why? ____________________)
      vi. Then it is necessary that $x \sim v$ (why? ____________________)
      vii. Thus we've shown $x \in$ ____ $\Rightarrow$ $x \in$ ____, and the subset proof is complete.

   (f) Proof of $\supseteq$: