Some warmups

A new definition: Let A be a subset of $\mathbb{R}$. Then A has a least element (say, x) if $x \leq y$ for all $y \in A$.

Exercise: Find the least element(s) of each of the following sets:

1. $S = \{x \mid x = 3n \text{ for some } n \in \mathbb{N}\}$
2. $T = \{x \mid x = 3n \text{ for some } n \in \mathbb{R}\}$
3. $U = \{x \in \mathbb{N} \mid x < 37\}$
4. $V = \{x \in \mathbb{Z} \mid x < 37\}$
5. $W = \{x \in \mathbb{Z} \mid 0 < x < 37\}$
6. $X = \{x \in \mathbb{R} \mid 0 < x < 37\}$
7. $Y = \{x \in \mathbb{N} \mid x \equiv 5 \text{ mod } 8\}$

Moral of the story?

Only subsets of $\mathbb{N}$ are guaranteed to have a least element.
This is an axiom, known as *The well-ordering property of* $\mathbb{N}$.

Another warmup

1. Let T be a subset of $\mathbb{N}$ that satisfies the following condition:

   $$\text{If } n \in T \text{ then } (n+1) \in T \quad (\text{for } n \in \mathbb{N})$$

   T is free to be anything it wants as long as it satisfies the above.

   Determine which of the following are examples of a valid T

   (a) $T_1 = \{6, 7, 8, \ldots\}$
   (b) $T_2 = \{6, 7, 8\}$
   (c) $T_3 = \{6, 8, 10, \ldots\}$
   (d) $T_4 = \{-5, -4, -3, \ldots\}$
   (e) $T_5 = \{6.5,\ 7.5,\ 8.5, \ldots\}$

2. Now, define $S = \mathbb{N} - T$ for some valid form of T. Some "easy" questions

   (a) Is $S \subseteq \mathbb{N}$?
   (b) Does S have a least element?
   (c) Suppose 35 is the least element of S. Which of the following are possible forms of T?
   $T = \{1, 2, 3, \ldots, 34\}$
   $T = \{36, 37, 38, \ldots\}$
   $T =$ any other set you can think of
   (d) Try the same experiment by assuming other possible values for the least element of S. E.g., 7 or 5 or 2 or 1. The goal is to determine whether it is possible to find a valid T for each case.

Moral of the story?
The only possible least element for S is 1.
In addition, S must be a finite set of the form: $\{1, 2, 3, \ldots, k\}$

The principle of mathematical induction

Theorem: Let T be a subset of $\mathbb{N}$ such that

(a) $1 \in T$; and

(b) If $n \in T$ then $(n+1) \in T$ (for all $n \in \mathbb{N}$)

Then $T = \mathbb{N}$.

Proof outline:

Let T be as described above.
By way of contradiction, suppose $T \neq \mathbb{N}$.
Then there is some natural number q that T does not contain. (Why?)
Let $S = T - \mathbb{N}$. Note that $q \in S$. (Why?)
Then, S is guaranteed to have a least element. (Why?)
Let m be this least element.
Then $(m-1)$ is in $\mathbb{N}$, (Why?) and $(m-1) \in T$. (Why?)
This implies $m \in T$. (Why?)
Thus, we have arrived at a contradiction. (Why?)
It follows that the principle of mathematical induction is true.

□