A warmup

Close your books and notes, and make a good-faith effort to define the following:

1. A relation
2. A reflexive relation

Be sure to include any needed context or subtext for your definition to be complete. Even if you don't get it 100% right, this is a great opportunity to sharpen your understanding!

Answers:

1. A relation from set A to set B is any subset of the Cartesian product $A \times B$. Not sure if "he" also wants us to define the Cartesian product $A \times B$, but here it is: $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$
2. A relation $\boldsymbol{r}$ on set A is reflexive if $(x, x) \in \boldsymbol{r}$ for each $x \in A$. This can also be written as $x \; \boldsymbol{r} \; x$ for each $x \in A$.

Sample proof of an equivalence relation

Example: Let $\boldsymbol{r}$ be a relation on $\mathbb{Z}$ defined by $x \,\boldsymbol{r}\, y$ iff $x \equiv y \bmod 7$. Prove that $\boldsymbol{r}$ is an equivalence relation.

Proof:

1. Let $\boldsymbol{r}$ be a relation on $\mathbb{Z}$ defined by $x \,\boldsymbol{r}\, y$ iff $x \equiv y \bmod 7$.
2. To prove $\boldsymbol{r}$ is an equivalence relation, we must show it is reflexive, symmetric, and transitive.
3. Proof of reflexive:
   (a) Let $x \in \mathbb{Z}$ [pick any element in the underlying set]
   (b) Then $x - x = 0 = 0 \cdot 7$ [by algebra]
   (c) This means $x \equiv x \bmod 7$. [defn. of congruence mod 7]
   (d) Since this holds for all $x \in \mathbb{Z}$, it follows that $\boldsymbol{r}$ is reflexive. [by defn. of reflexive]
4. Proof of symmetry:
   (a) Let $x, y \in \mathbb{Z}$ such that $x\boldsymbol{r}y$. [pick any 2 related elements in underlying set]
   (b) Then $x - y = 7k$ for some $k \in \mathbb{Z}$. [by defn. of $\boldsymbol{r}$]
   (c) This means $-(x - y) = -7k$. [by algebra]
   (d) Thus, $y - x = 7n$ where $n = -k$ is in $\mathbb{Z}$. [by algebra]
   (e) Therefore, $y \equiv x \bmod 7$, which means $y\boldsymbol{r}x$.
   (f) It follows that $\boldsymbol{r}$ is symmetric. [since $x\boldsymbol{r}y \Rightarrow y\boldsymbol{r}x$ for all $x, y \in \mathbb{Z}$]
5. Proof of transitive:
   (a) Let $x, y, z \in \mathbb{Z}$ such that $x\boldsymbol{r}y$ and $y\boldsymbol{r}z$. [pick any 3 pairwise related elements in underlying set]
   (b) Then, $x - y = 7k$ and $y - z = 7n$ for some $k, n \in \mathbb{Z}$.
   ... [students complete the story]

Some explorations

Equivalence relations have some amazing properties that make them very useful in many areas of mathematics.

Let's explore some of these using the relation congruence mod 2 on $\mathbb{Z}$. Here are some exercises:

1. Find all the elements of $\mathbb{Z}$ related to 0. Let's name this set z_0.
2. Find all the elements related to 1. Call this set z_1.
3. Find all the elements related to 2. Call this set z_2.

   $\vdots$

Keep doing this till you notice some pattern.
What connections do you see between the sets z_0, z_1, z_2, etc?
Write your thoughts/observations on each of the following questions:

1. Are z_0, z_1, z_2, etc, subsets of $\mathbb{Z}$?
2. What is $z_0 \cap z_1$, $z_1 \cap z_2$, etc?
3. How about $z_0 \cap z_2$, $z_1 \cap z_3$, etc?
4. What is $z_0 \cup z_1 \cup z_2 \cup \ldots$?

Moral of the story: Equivalence classes