Student name:

Earlham College

MATH 288: Sophomore Seminar **Test 1 - In class part**

Spring 2012 **February 23, 2012**

Instructions:

- Answer all questions on separate paper (not on this sheet!).
- This part is a regular "closed-book" test, and is to be taken without the use of notes, books, or other reference materials.
- This portion of the test adds up to 24 points.

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(1) [3 pts. each$\times 2 = 6$ pts.] Give a mathematically precise definition for the following terms. Include any context needed for your definition to make sense.

(a) Equivalence class.

(b) Union over an arbitrary collection of sets.

For reference, here is an example of a mathematically precise definition (of relative complement): Let S and T be sets. Then the relative complement of T in S is the set $S - T = \{x : x \in S \text{ and } x \notin T\}$

(2) [2 pts. each$\times 3 = 6$ pts.] Negate the following statements. Negation must be written in complete words and sentences, with sparing use of symbolism. (Yes, you may use the standard arithmetic symbols used in the questions themselves –, =, ≤, etc.)

(a) If a quadrilateral is divided into four triangles, then each triangle has equal area.

(b) There exist integers q and r such that for all natural numbers m, $m = q + r$.

(c) For all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $f(z) > 0$ whenever $z > y$.

(3) [2 pts. each$\times 3 = 6$ pts.] Prove or give a counterexample for each of the following propositions. Your solution must show how your proof, example or counterexample does the job.

(a) There exists an $x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}$, $y^2 \neq x$.

(b) For all sets A, B and C, if $(A \cup C) \subseteq (B \cup C)$, then $A \subseteq B$.

(c) The relation $r = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$ is an equivalence relation on the set $A = \{1,2,3\}$.

(4) [3 pts. each$\times$2 $=$ 6 pts.] Write the opening lines of a proof by contrapositive for each of the following (not necessarily true!) propositions. See the example below for clarification on what "opening lines" means.

(a) For all sets A, B and C, if $A \subseteq B$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.

(b) For all sets A, B, C and D, if $(A \times C) \not\subseteq (B \times D)$, then $A \not\subseteq B$ or $C \not\subseteq D$.

Opening lines must include: contrapositive statement, what hypotheses you will assume, what conclusion you will prove, and begin the element argument.
Example: Prove the proposition "For all sets S and T, if $(S \cap T) = S$, then $(S \cup T) = T$."
Proof (opening lines):
(1) {* *What type of proof*} I will prove this via the contrapositive.
Contrapositive statement: For all sets S and T, if $(S \cup T) \neq T$, then $(S \cap T) \neq S$.
(2) {* *Hypotheses*} Let S and T be sets such that $(S \cup T) \neq T$.
(3) {* *I'll show*} I will prove $(S \cap T) \neq S$ by showing $(S \cap T) \not\subseteq S$ or $S \not\subseteq (S \cap T)$.
(4) {* *Start element argument*} Since $(S \cup T) \neq T$ there exists $m \in S \cup T$ such that $m \notin T$.
(5) Then ...

End of test

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MATH 288: Sophomore Seminar: Spring 2012

Take home portion of Test 1 – administered via the Science Library.
Test must be taken between February 23-27 (by 2:30 PM).
Complete and return to Science Library within 24 hours after checkout, or before library closing time, whichever comes first.

Instructions:

- Answer all questions on separate paper – not on this sheet!
- You may use the following reference materials: The textbook, your own class notes and homework, supplementary handouts given in class, any materials posted on the class website that were prepared for this class.
- Prohibited materials: Any other reference sources, including electronic, printed, written or verbal.
- This portion of the test adds up to 24 points – 8 points for each question.
  Total for in-class+take-home = 24+24 = 48.

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(1) Let A and B be sets. Prove that if $B^c \subseteq A^c$, then $A \cup B = B$.

(2) Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\Gamma}$ be an indexed collection of sets, and let C be a set. Prove that $C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha \right) = \bigcup_{\alpha\in\Gamma} (C \cap B_\alpha)$.

(3) Let S be a set, and let r be any relation on S (not necessarily an equivalence relation). For each $x \in S$ we can define its set of relatives, say $[x]$, as follows

$$[x] = \{y \in S : (x, y) \in \mathsf{r}\}$$

Suppose $S = \{a, b, c, d, e, f\}$, and
$\mathsf{r} = \{(a, a), (b, b), (b, c), (b, e), (c, e), (d, b), (d, c), (d, e), (e, b), (e, c), (e, e)\}$

(a) Determine whether r is reflexive, symmetric or transitive. Prove your claims.

(b) For each $x \in S$ find $[x]$.

(c) Give an example of an equivalence relation on the given set S. Justify/prove your claim.

End of test

Soph. Sem: Spring 2012: Solution to Test 1

In-class part

(1) Give a mathematically precise definition for the following terms. Include any context needed for your definition to make sense.

(a) Equivalence class.
Solution: Let r be an equivalence relation on set A. Then the equivalence class of any $x \in A$ is given by the set

$$[x] = \{y \in A : x \mathrel{\mathsf{r}} y\}$$

(b) Union over an arbitrary collection of sets.
Solution: Let $\mathcal{A} = \{A_\gamma : \gamma \in \Gamma\}$ be a collection of sets, where Γ denotes some index set. Then the union over the collection is the set

$$\bigcup_{\gamma \in \Gamma} A_\gamma = \{x : x \in A_\gamma \text{ for some } \gamma \text{ in } \Gamma\}$$

(2) Negate the following statements:

(a) If a quadrilateral is divided into four triangles, then each triangle has equal area.
Solution: There exists a quadrilateral that, when divided into four triangles, gives at least two triangles whose areas differ.

(b) There exist integers q and r such that for all natural numbers m, $m = q + r$.
Solution: For all integers q and r there exists natural number m such that $m \neq q + r$.

(c) For all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $f(z) > 0$ whenever $z > y$.
Solution: There exists a real number x such that for every real number y, $f(z) \leq 0$ for some $z > y$.

(3) Prove or give a counterexample for each of the following propositions.

(a) There exists an $x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}$, $y^2 \neq x$.
Solution: This is true. Here is a proof: Let $x = -2$. By way of contradiction, suppose there is some $y \in \mathbb{Z}$ such that $y^2 = x$. This would imply $y^2 = -2$, which requires $y = \sqrt{-2}$. This is a contradiction, because $\sqrt{-2} \notin \mathbb{Z}$.

(b) For all sets A, B and C, if $(A \cup C) \subseteq (B \cup C)$, then $A \subseteq B$.
Solution: This is false. Here is a counterexample: Let $A = \{2\}$, $B = \{3\}$ and $C = \{2, 3\}$. Then $(A \cup C) = (B \cup C) = \{2, 3\}$. Therefore, $(A \cup C) \subseteq (B \cup C)$, but $A \nsubseteq B$.

(c) The relation $r = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$ is an equivalence relation on the set $A = \{1, 2, 3\}$.
Solution: This is true. Proof: For each $x \in A$, we see that $(x, x) \in \mathsf{r}$. Therefore, r is reflexive. Next, consider any $x, y \in A$ such that $(x, y) \in \mathsf{r}$. We see that $(y, x) \in \mathsf{r}$. For instance, $(2, 3) \in \mathsf{r} \Rightarrow (3, 2) \in \mathsf{r}$. Therefore, it is symmetric.
It is transitive because for all $x, y, z \in A$, if $(x, y) \in \mathsf{r}$ and $(y, z) \in \mathsf{r}$ then $(x, z) \in \mathsf{r}$.
Since r is reflexive, symmetric and transitive it is an equivalence relation on A.

(4) Write the opening lines of a proof by contrapositive for each of the following (not necessarily true!) propositions.

(a) For all sets A, B and C, if $A \subseteq B$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.

Solution:

(1) I will prove this via the contrapositive.

Contrapositive: If $A \cap C \neq \emptyset$, then $A \nsubseteq B$ or $B \cap C \neq \emptyset$.

(2) Let A, B and C be sets such that $A \cap C \neq \emptyset$.

(3) I will prove $A \nsubseteq B$ or $B \cap C \neq \emptyset$.

(4) To this end, suppose $A \subseteq B$. I'll show that $B \cap C \neq \emptyset$.

(5) Since $A \cap C \neq \emptyset$, let $x \in A \cap C$. ...

(b) For all sets A, B, C and D, if $(A \times C) \nsubseteq (B \times D)$, then $A \nsubseteq B$ or $C \nsubseteq D$.

Solution:

(1) I will prove this via the contrapositive.

Contrapositive: If $A \subseteq B$ and $C \subseteq D$, then $(A \times C) \subseteq (B \times D)$.

(2) Let A, B, C and D be sets such that $A \subseteq B$ and $C \subseteq D$.

(3) I will prove $(A \times C) \subseteq (B \times D)$.

(4) Let $(m, n) \in (A \times C)$.

(5) Then ...

Soph. Sem: Spring 2012: Solution to Test 1

Take-home part

(1) Let A and B be sets. Prove that if $B^c \subseteq A^c$, then $A \cup B = B$.

Solution:

(1) I will prove this directly.
(2) Let A, B and C be sets such that $B^c \subseteq A^c$.
(3) I will prove $A \cup B = B$ by showing $A \cup B \subseteq B$ and $B \subseteq A \cup B$.
(4) To prove $A \cup B \subseteq B$:
 (4.1) Let $m \in A \cup B$.
 (4.2) Then $m \in A$ or $m \in B$. [definition of union]
 (4.3) If $m \in B$ we are done, since $m \in A \cup B \Rightarrow m \in B$.
 (4.4) Otherwise, consider $m \in A$.
 (4.5) This means $m \notin A^c$. [negation of complement definition]
 (4.6) Then $m \notin B^c$. [since $B^c \subseteq A^c$ by hypothesis]
 (4.7) This implies $m \in B$.
 (4.8) Lines (4.1), (4.3) and (4.7) show $m \in A \cup B \Rightarrow m \in B$.
  Therefore, $A \cup B \subseteq B$ [definition of subset]
(5) To prove $B \subseteq A \cup B$:
 (5.1) Let $n \in B$.
 (5.2) Then $n \in A \cup B$. [definition of union]
 (5.3) So we have: $n \in B \Rightarrow n \in A \cup B$.
  Therefore, $B \subseteq A \cup B$ [definition of subset]
(6) It follows from (4) and (5) that $A \cup B = B$. [definition of set equality]
(7) Therefore, we have proved: $B^c \subseteq A^c \Rightarrow A \cup B = B$.

(2) Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\Gamma}$ be an indexed collection of sets, and let C be a set. Prove that

$$C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right) = \bigcup_{\alpha\in\Gamma}(C \cap B_\alpha).$$

Solution:

(1) I will prove this by showing
 $C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right) \subseteq \bigcup_{\alpha\in\Gamma}(C \cap B_\alpha)$ and $\bigcup_{\alpha\in\Gamma}(C \cap B_\alpha) \subseteq C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right)$.
(2) Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\Gamma}$ be an indexed collection of sets, and let C be a set.
(3) Proof of $C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right) \subseteq \bigcup_{\alpha\in\Gamma}(C \cap B_\alpha)$:
 (3.1) Let $x \in C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right)$.
 (3.2) Then $x \in C$ and $x \in \bigcup_{\alpha\in\Gamma} B_\alpha$. [definition of intersection]
 (3.3) $x \in \bigcup_{\alpha\in\Gamma} B_\alpha$ implies $x \in B_\alpha$ for some $\alpha \in \Gamma$. [definition of union]
 (3.4) For some $\alpha \in \Gamma$: $x \in C$ by line (3.2), and $x \in B_\alpha$ by line (3.3).
 (3.5) It follows that $x \in C \cap B_\alpha$, for that particular α.
 (3.6) Then $x \in \bigcup_{\alpha\in\Gamma}(C \cap B_\alpha)$. [definition of union]
 (3.7) Lines (3.1) and (3.6) show $x \in C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right) \Rightarrow x \in \bigcup_{\alpha\in\Gamma}(C \cap B_\alpha)$.
  Therefore, $C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right) \subseteq \bigcup_{\alpha\in\Gamma}(C \cap B_\alpha)$ [definition of subset]
(4) Proof of $\bigcup_{\alpha\in\Gamma}(C \cap B_\alpha) \subseteq C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right)$:
 (4.1) Let $s \in \bigcup_{\alpha\in\Gamma}(C \cap B_\alpha)$.

(4.2) Then, for some $\alpha \in \Gamma$, $s \in (C \cap B_\alpha)$. [definition of union]
(4.3) For that α, $s \in C$ and $s \in B_\alpha$. [definition of intersection]
(4.4) This implies $s \in C$ and $s \in B_\alpha$ for some $\alpha \in \Gamma$.
(4.5) Thus $s \in C$ and $s \in \bigcup_{\alpha\in\Gamma} B_\alpha$. [definition of union]
(4.6) It follows that $s \in C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right)$. [definition of intersection]
(4.7) Lines (4.1) and (4.6) show $s \in \bigcup_{\alpha\in\Gamma}(C \cap B_\alpha) \Rightarrow s \in C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right)$.
Therefore, $\bigcup_{\alpha\in\Gamma}(C \cap B_\alpha) \subseteq C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right)$ [definition of subset]

(5) It follows from (3) and (4) that $C \cap \left(\bigcup_{\alpha\in\Gamma} B_\alpha\right) = \bigcup_{\alpha\in\Gamma}(C \cap B_\alpha)$.
[definition of set equality]

(3) Let S be a set, and let r be any relation on S (not necessarily an equivalence relation). For each $x \in S$ we can define its set of relatives, say $[x]$, as follows

$$[x] = \{y \in S : (x, y) \in \mathsf{r}\}$$

Suppose $S = \{a, b, c, d, e, f\}$, and
$\mathsf{r} = \{(a, a), (b, b), (b, c), (b, e), (c, e), (d, b), (d, c), (d, e), (e, b), (e, c), (e, e)\}$

(a) Determine whether r is reflexive, symmetric or transitive. Prove your claims.

Solution:

r is not reflexive, since the elements c, d and f of A are not related to themselves. Reflexivity requires that $(x, x) \in \mathsf{r}$ for each $x \in A$.

r is not symmetric, since $(b, c) \in \mathsf{r}$ but $(c, b) \notin \mathsf{r}$. Similarly, $(d, b) \in \mathsf{r}$ but $(b, d) \notin \mathsf{r}$. Symmetry requires that $(x, y) \in \mathsf{r} \Rightarrow (y, x) \in \mathsf{r}$.

r is not transitive, since $(c, e) \in \mathsf{r}$ and $(e, b) \in \mathsf{r}$, but $(c, b) \notin \mathsf{r}$.
Transitivity requires: $(x, y) \in \mathsf{r}$ and $(y, z) \in \mathsf{r} \Rightarrow (x, z) \in \mathsf{r}$.

(b) For each $x \in S$ find $[x]$.

Solution:

$[a] = \{a\}$
$[b] = \{b, c, e\}$
$[c] = \{e\}$
$[d] = \{b, c, e\}$
$[e] = \{b, c, e\}$
$[f] = \emptyset$

(c) Give an example of an equivalence relation on the given set S. Justify/prove your claim.

Solution:

Let $\mathsf{r} = \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f)\}$
Then r is an equivalence relation on A.
Justification:
Clearly r is reflexive, since $(x, x) \in \mathsf{r}$ for each $x \in A$.
r also satisfies the definition of symmetric and transitive, since implications are vacuously true whenever their hypotheses are false.
Thus, r is an equivalence relation on A.