Miscellaneous practice worksheet

1. Let S be a set, and let r be any relation on S (not necessarily an equivalence relation). For each $x \in S$ we can define its set of relatives, say $[x]$, as follows

$$[x] = \{y \in S : (x, y) \in \mathsf{r}\}$$

Suppose $S = \{a, b, c, d, e, f\}$, and
$\mathsf{r} = \{(a, a), (b, b), (b, c), (b, e), (c, e), (d, b), (d, c), (d, e), (e, b), (e, c), (e, e)\}$

(a) Determine whether r is reflexive, symmetric or transitive. Prove your claims.

(b) For each $x \in S$ find $[x]$.

(c) Give an example of an equivalence relation on the given set S. Justify/prove your claim.

(*From spring '12, Test 1*)

2. Determine whether each of the following functions is injective, surjective, neither, or both. Prove your claims.

   (a) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $f(m, n) = (n, m)$.

   (b) $g : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $g(t) = (t, t)$.

(*From spring '12, Test 2*)

3. Write a statement negating each of the following statements. Negation must be written in complete words and sentences, with sparing use of symbolism (except for symbols used in the question itself).

   (a) Let A and B be sets, and let $f : A \to B$ be a function. If f is injective then it has a left inverse.

   (b) For all sets A, B and C, if $C \cap (A \cap B) = \emptyset$, then $C \cap A = \emptyset$ or $C \cap B = \emptyset$.

   (c) There exists a subset S of $\mathbb{N}$ such that for all $x \in S$, $x > y$ for some $y \in S$.

(*From spring '12, final exam*)

4. This question deals with images and pre-images, and is therefore grouped together. Other than that, the individual parts of the question are not directly connected to each other.

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5 - 3x$. Let $T = \{y \in \mathbb{R} : |y - 2| < 1\}$. Find $f^{-1}(T)$ and $f(f^{-1}(T))$.

   (b) Give an example of a function $f : A \to B$ and $T \subseteq B$ such that $f(f^{-1}(T)) \neq T$. Be sure to clearly define your sets A, B and T, and to show $f(f^{-1}(T)) \neq T$.

   (c) Same as question (b), except find $S \subseteq A$ such that $f^{-1}(f(S)) \neq S$.

(*From spring '12, Test 2*)

5. Here is a new definition: Let A be a set, and let $f : A \to A$ and $g : A \to A$ be functions. The functions f and g are said to *commute* if $g \circ f = f \circ g$. Let f, g and h be functions from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = -x$, $g(x) = x^n$ and $h(x) = ax + b$. Here $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$.

   (a) Determine a condition on n that will guarantee that f and g commute. Justify or prove your claim.

   (b) Show that f and h commute if and only if $b = 0$.

   (*From spring '12, Test 2*)

6. Prove the following statement

$$3 + 11 + \cdots + (8n - 5) = 4n^2 - n \quad \text{for all} \quad n \in \mathbb{N}$$

   (*From spring '12, final exam*)

7. Give a mathematically precise definition for the following terms. Include any context needed for your definition to make sense.

   (a) An injective function.

   (b) A surjective function.

   (c) The inverse of a function.

   (d) A partition (of a set).

   (*From misc exams*)

8. Let $f : A \to B$ be an injective function. Define a function $g : A \to f(A)$ by $g(x) = f(x)$ for all $x \in A$. Show that g is bijective.

9. Prove that $2^{n+1} < n!$ for all natural numbers larger than some natural number k.

10. Let $\mathtt{r}$ be a relation on $\mathbb{Z}$ defined by $x \ \mathtt{r} \ y$ iff $|x - y| \leq 3|$. Determine whether $\mathtt{r}$ is reflexive, symmetric, and transitive.

11. Let $S = \mathbb{N} \times \mathbb{N}$ and suppose $\mathtt{r}$ is a relation on S defined by $(u, v)\mathtt{r}(x, y)$ iff $u^v = x^y$.

    a. Show that $\mathtt{r}$ is an equivalence relation on S.
    b. Find the equivalence class $E_{(9,2)}$.
    c. Find an equivalence class with exactly one element.
    d. Find an equivalence class with exactly two elements.

12. Let $\mathcal{B} = \{B_\alpha\}_{\alpha \in \Gamma}$ be an indexed collection of sets, and let C be a set. Prove that $C \cap \left(\bigcup_{\alpha \in \Gamma} B_\alpha \right) = \bigcup_{\alpha \in \Gamma} (C \cap B_\alpha)$.

    (*From spring '12, Test 1*)

13. One of the exercises in the textbook proves that if $f : A \to B$, and $C, D \subseteq A$, then $f(C \cap D) \subseteq f(C) \cap f(D)$. Prove the following generalization of this result:

(a) Let $f : A \to B$ be a function from set A to B. Let $\{C_j\}$ for $j \in J$ be a collection of subsets of A. Then $f(\bigcap_{j \in J} C_j) \subseteq \bigcap_{j \in J} f(C_j)$.

(b) Try to prove $\bigcap_{j \in J} f(C_j) \subseteq f(\bigcap_{j \in J} C_j)$ and explain where/why the proof fails.

14. Let A and B be sets and let $f : A \to B$ be a function. Suppose $C \subseteq A$ and $D \subseteq B$. Prove or disprove each of the following
(a) If $f(C) \subseteq D$, then $C \subseteq f^{-1}(D)$.
(b) If $C \subseteq f^{-1}(D)$, then $f(C) \subseteq D$. *(From spring '12, final exam)*

15. Let $\mathtt{r}$ and $\mathtt{s}$ be equivalence relations on some set A.
(a) Show that the relation $\mathtt{r} \cup \mathtt{s}$ is reflexive and symmetric on A.
(b) Show that $\mathtt{r} \cup \mathtt{s}$ need not be transitive on A.
[Hint: Denote the relation $\mathtt{r} \cup \mathtt{s}$ as $\mathtt{u}$, say. What would make $\mathtt{u}$ reflexive, symmetric or transitive on A? How does that relate to what you know about $\mathtt{r}$ and $\mathtt{s}$?]

(From spring '12, final exam)