1.15 PRACTICE Complete the following proof of Theorem 1.13(f).

Proof: We wish to prove that $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$. To this end, let $x \in A \backslash (B \cup C)$. Then ___________ and ____________. Since $x \notin B \cup C$, __________ and $x \notin C$ (for if it were in either B or C, then it would be in their union). Thus $x \in A$ and $x \notin B$ and $x \notin C$. Hence $x \in A \backslash B$ and $x \in A \backslash C$, which implies that ____________. We conclude that $A \backslash (B \cup C) \subseteq (A \backslash B) \cap (A \backslash C)$.

Conversely, suppose that $x \in$ _______________. Then $x \in A \backslash B$ and $x \in A \backslash C$. But then $x \in$ _______ and $x \notin$ _______ and $x \notin$ _______. This implies that $x \notin (B \cup C)$, so $x \in$ ________________. Hence _______________ $\subseteq$ _______________ as desired. ♦

Up to this point we have talked about combinations of two or three sets. By repeated application of the appropriate definitions we can even consider unions and intersections of any finite collection of sets. But sometimes we want to deal with combinations of infinitely many sets, and for this we need a new notation and a more general definition.

1.16 DEFINITION If for each element j in a nonempty set J there corresponds a set A_j, then

$$\mathscr{A} = \{A_j : j \in J\}$$

is called an **indexed family** of sets with J as the index set. The union of all the sets in $\mathscr{A}$ is defined by

$$\bigcup\nolimits_{j \in J} A_j = \{x : x \in A_j \text{ for some } j \in J\}.$$

The intersection of all the sets in $\mathscr{A}$ is defined by

$$\bigcap\nolimits_{j \in J} A_j = \{x : x \in A_j \text{ for all } j \in J\}.$$

Other notations for $\bigcup\nolimits_{j \in J} A_j$ include $\bigcup\limits_{j \in J} A_j$ and $\bigcup \mathscr{A}$.

If $J = \{1, 2, \ldots, n\}$, we may write

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{j=1}^{n} A_j \quad \text{or} \quad \bigcup\nolimits_{j=1}^{n} A_j,$$

and if $J = \mathbb{N}$, the common notation is

$$\bigcup_{j=1}^{\infty} A_j \quad \text{or} \quad \bigcup\nolimits_{j=1}^{\infty} A_j.$$

Similar variations of the notation apply to intersections.

There are some situations where a family of sets has not been indexed but we still wish to take the union or intersection of all the sets. If $\mathscr{B}$ is a nonempty collection of sets, then we let

$$\bigcup_{B \in \mathscr{B}} B = \{x : x \in B \text{ for some } B \in \mathscr{B}\}$$

and

$$\bigcap_{B \in \mathscr{B}} B = \{x : x \in B \text{ for all } B \in \mathscr{B}\}.$$

1.17 EXAMPLE For each $k \in \mathbb{N}$, let $A_k = [0, 2 - 1/k]$. Then $\bigcup_{k=1}^{\infty} A_k = [0, 2)$.

1.18 PRACTICE Let $S = \{x \in \mathbb{R} : x > 0\}$. For each $x \in S$, let $A_x = (-1/x, 1/x)$. Find $\bigcap_{x \in S} A_x$.

Review of Key Terms in Section 1

Subset	Empty set	Disjoint sets
Proper subset	Union	Indexed family
Equal sets	Intersection	
Interval	Complement	

ANSWERS TO PRACTICE PROBLEMS

1.2 (a), (b), and (d) are sets. (c) is not a set unless “tall” and “in” are made precise.

1.6 (a), (c), (f), (g), and (h) are true.

1.9 $x \in A \Rightarrow x \in B$.

1.11 If $x \in A \cap (U \setminus B)$, then $x \in A$ and $x \in \underline{U \setminus B}$, by the definition of intersection. But $x \in U \setminus B$ means that $x \in U$ and $\underline{x \notin B}$. Since $x \in A$ and $x \notin B$, we have $x \in \underline{A \setminus B}$, as required. Thus $A \cap (U \setminus B) \subseteq A \setminus B$.

Conversely, we must show that $\underline{A \setminus B} \subseteq \underline{A \cap (U \setminus B)}$. If $\underline{x \in A \setminus B}$, then $x \in A$ and $x \notin B$. Since $A \subseteq U$, we have $x \in \underline{\;U\;}$. Thus $x \in U$ and $x \notin B$, so $\underline{x \in U \setminus B}$. But then $\underline{x \in A}$ and $x \in U \setminus B$, so $x \in A \cap (U \setminus B)$. Hence $A \setminus B \subseteq A \cap (U \setminus B)$. ♦

2.12 DEFINITION A **partition** of a set S is a collection $\mathscr{P}$ of nonempty subsets of S such that

(a) Each $x \in S$ belongs to some subset $A \in \mathscr{P}$.
(b) For all $A, B \in \mathscr{P}$, if $A \neq B$, then $A \cap B = \varnothing$.

A member of $\mathscr{P}$ is called a **piece** of the partition.

2.13 EXAMPLE Let $S = \{1, 2, 3\}$. Then the collection $\mathscr{P} = \{\{1\}, \{2\}, \{3\}\}$ is a partition of S. We may picture this as in Figure 4(a). The collection $\mathscr{P} = \{\{1, 2\}, \{3\}\}$ is also a partition of S. [See Figure 4(b).] But the collection $\mathscr{P} = \{\{1, 2\}, \{2, 3\}\}$ is not a partition of S because $\{1, 2\}$ and $\{2, 3\}$ are not disjoint.

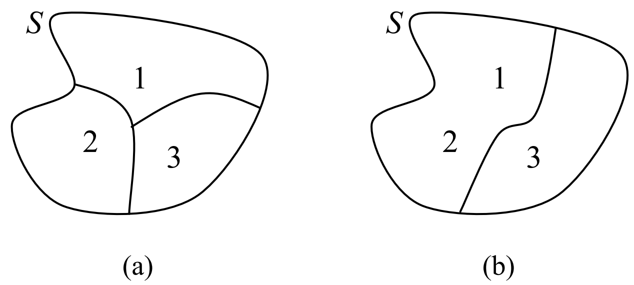

Figure 4 Two partitions of set $S = \{1, 2, 3\}$

2.14 PRACTICE Is $\mathscr{P} = \{\{1, 3\}, \{2\}\}$ a partition of $S = \{1, 2, 3, 4\}$?

2.15 EXAMPLE Let S be the set of all students in a particular university. For x and y in S, define $x\mathsf{R}y$ iff x and y were born in the same calendar year. Then R is an equivalence relation, and a typical equivalence class is the set of all students who were born in a particular year. For example, if student x was born in 1992, then E_x consists of all the students who were born in 1992. This relation partitions S into disjoint subsets, where students born in the same year are grouped together.

2.16 PRACTICE In Example 2.15, if $y \in E_x$, does this mean that x and y are the same age?

Not only does an equivalence relation on a set S determine a partition of S, but the partition can be used to determine the relation. We formalize this in the following theorem.

2.17 THEOREM Let R be an equivalence relation on a set S. Then $\{E_x : x \in S\}$ is a partition of S. The relation "belongs to the same piece as" is the same as R. Conversely, if $\mathscr{P}$ is a partition of S, let P be defined by $x\mathsf{P}y$ iff x and y are in the same piece of the partition. Then P is an equivalence relation and the corresponding partition into equivalence classes is the same as $\mathscr{P}$.