A warmup

Recall: Neither order, nor duplication of elements makes any difference to a set. In other words,

$$\{0, 3\} = \{3, 3, 3, 0, 3, 0, 0, 3\}, \text{ etc.}$$

But, that is a problem when we want to create mathematical structures where order and duplication both matter, such as ordered pairs. With that in mind, here is a set theoretic definition of an ordered pair

$$(x, y) = \{\{x\}, \{x, y\}\}$$

It can be proved that: $(x, y) = (p, q)$ iff $x = p$ and $y = q$.
This implies: $(x, y) = (y, x)$ iff $x = y$. [Q: Can you prove it?]

Exercise: Some clever mathematician came up with the above idea for ordered pairs. Unfortunately, it does NOT generalize to ordered triples. For example, it doesn't work if we were to define

$$(x, y, z) = \{\{x\}, \{x, y\}, \{x, y, z\}\}$$

To show that it fails, find two different ordered triples whose set theoretic definition looks the same.

Proofs involving Cartesian products

Example: For all sets A, B, C, $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

Proof:

1. Let A, B, C be sets.
2. To prove equality, we must show subset both ways.
3. Proof of $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$.
   (a) Let $(x, y) \in (A \cap B) \times C$. [pick any element in cross product]
   (b) Then $x \in (A \cap B)$ and $y \in C$. [defn. of cross product]
   (c) This means $x \in A$ and $x \in B$. [defn. of $A \cap B$]
   (d) Thus, $(x, y) \in A \times C$ and $(x, y) \in B \times C$.
   [since $y \in C$ and by defn. of cross product]
   (e) Then $(x, y) \in (A \times C) \cap (B \times C)$
   [from line (d), by defn. of intersection]
   (f) Thus, we've shown $(x, y) \in (A \cap B) \times C \Rightarrow$
   $(x, y) \in (A \times C) \cap (B \times C)$.
   (g) It follows that $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$.
4. Next, we show that $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$.
   (a) Let ... [complete the story]

Some "easy" exercises

1. Let $S = \{1, 2\}$, $T = \{l, m, n\}$.
   Find $S \times S$ and $T \times S$.

2. Show that $(a, a) = \{\{a\}\}$.

3. Let $S = \{3\}$ and $T = \{l, m, n\}$.
   How many possible relations exist from S to T?
   List all of them in correct set theoretic notation.

4. Let S be the set of students in this class, and C be the set of chairs in this classroom. Define a relation from S to C.