General template for writing proofs in set theory

Proof:

{ State the proof strategy you will use – direct, by contradiction, by contrapositive, etc. If proving by contrapositive, restate the the proposition as you will prove it}*

{ State your hypotheses}*

{ State your contradictory supposition, or state what you propose to show}*

{ Pick a suitable element to start your element argument}*

{ Use logic, together with hypotheses, to trace your element's properties}*

{ Tie your results together in a concluding statement, that states what you have shown}*

Some examples

Exercise 2.3.6: Let A, B, and X be sets. If $(A \cup B) \subseteq X$ and $(X - B) \subseteq (X - A)$, then $A \subseteq B$.

Solution: We will use a proof by contradiction.

{ State your hypotheses}*

(1) Let A, B, and X be sets such that $(A \cup B) \subseteq X$ and $(X - B) \subseteq (X - A)$.

{ State your contradictory supposition}*

(2) By way of contradiction, suppose $A \nsubseteq B$.

{ Pick a suitable element to start your element argument}*

(3) Since $A \nsubseteq B$, there exists $p \in A$ such that $p \notin B$. [by negating definition of subset]

{ Use logic, together with hypotheses, to trace your element's properties}*

(4) Because $p \in A$, we know $p \in A \cup B$. [by definition of union]

(5) By our hypotheses, this implies $p \in X$. [because $(A \cup B) \subseteq X$]

(6) Lines (5) and (3) imply $p \in X$ and $p \notin B$. Thus $p \in X - B$. [definition of complement]

(7) Since $p \in X - B$, we conclude $p \in X - A$. [since $(X - B) \subseteq (X - A)$ by hypothesis]

(8) $p \in X - A \Rightarrow p \in X$ and $p \notin A$. [definition of complement]

(9) This contradicts line (3): $p \in A$ and $p \notin A$.

{ Tie your results together in a concluding statement}*

(10) Therefore, we have shown $A \nsubseteq B$ leads to a contradiction. It follows that if $(A \cup B) \subseteq X$ and $(X - B) \subseteq (X - A)$, then $A \subseteq B$.

Exercise 2.4.2: Let A, B, and X be sets. Then $(X - A) \cup (X - B) \subseteq X - (A \cap B)$.
Solution: We will use a direct proof.
{* *State your hypotheses*}
(1) Let A, B, and X be sets.
{* *State what you propose to prove*}
(2) I will show that $(X - A) \cup (X - B) \subseteq X - (A \cap B)$.
{* *Pick a suitable element to start your element argument*}
(3) Let q be any element of the set $(X - A) \cup (X - B)$.
This means we're assumming it is non-empty – otherwise there is nothing to prove!
{* *Use logic, together with hypotheses, to trace your element's properties* }
(4) $q \in (X - A) \cup (X - B) \Rightarrow q \in (X - A)$ or $q \in (X - B)$. [by definition of union]
(5) We consider each of the two possibilities separately
(5.1) $q \in (X - A)$.
(5.1.1) $q \in (X - A) \Rightarrow q \in X$ and $q \notin A$. [definition of complement]
(5.1.2) $q \notin A \Rightarrow q \notin (A \cap B)$. [by negating definition of intersection]
(5.1.3) From (5.1.1) and (5.1.2): $q \in X$ and $q \notin (A \cap B)$.
Therefore, $q \in X - (A \cap B)$. [by definition of complement]
(5.2) $q \in (X - B)$. By similar argument to (5.1) we have
(5.2.1) $q \in X$ and $q \notin B \Rightarrow q \notin (A \cap B) \Rightarrow q \in X - (A \cap B)$.
{* *Tie your results together in a concluding statement*}
(6) Therefore, we've shown $q \in (X - A) \cup (X - B) \Rightarrow q \in X - (A \cap B)$ for every case.
It follows from the definition of subset that $(X - A) \cup (X - B) \subseteq X - (A \cap B)$.

Exercise 2.6.3: For all sets A, B, C and D, if $A \subseteq C$ and $B \subseteq D$ then $(A \cup B) \subseteq (C \cup D)$.
Solution: We will use a direct proof.
{* *State your hypotheses*}
(1) Let A, B, C, and D be sets such that $A \subseteq C$ and $B \subseteq D$.
{* *State what you propose to prove*}
(2) I will show that $(A \cup B) \subseteq (C \cup D)$.
{* *Pick a suitable element to start your element argument*}
(3) Let s be any element of the set $A \cup B$ (assumed non-empty – otherwise proof is vacuous).
{* *Use logic, together with hypotheses, to trace your element's properties* }
(4) $s \in (A \cup B) \Rightarrow s \in A$ or $s \in B$. [by definition of union]
(5) We consider each of the two possibilities separately
(5.1) Suppose $s \in A$.
(5.1.1) Then $s \in C$. [by hypothesis $A \subseteq C$, and by definition of subset]
(5.1.2) This implies $s \in (C \cup D)$. [by definition of union]
(5.2) Suppose $s \in B$. By similar argument to (5.1) we have
(5.2.1) $s \in D$ because $B \subseteq D$, which implies $s \in (D \cup C)$ by definition of union.
{* *Tie your results together in a concluding statement*}
(6) Therefore, we've shown $s \in (A \cup B) \Rightarrow s \in (C \cup D)$ for every case.
It follows from the definition of subset that $(A \cup B) \subseteq (C \cup D)$.