13. Consider

$$m\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = 0.$$

That is, fix $b = 1$ and $k = 2$, and let $0 < m < \infty$.

14. Using the `DETools` program `TDPlaneQuiz`, describe the path through the trace-determinant plane that was used to produce each animation.

3.8 LINEAR SYSTEMS IN THREE DIMENSIONS

So far, we have studied linear systems with two dependent variables. For these systems, the behavior of solutions and the nature of the phase plane can be determined by computing the eigenvalues and eigenvectors of the 2×2 coefficient matrix. Once we have found two solutions with linearly independent initial conditions, we can give the general solution.

In this section we show that the same is true for linear systems with three dependent variables. The eigenvalues and eigenvectors of the 3×3 coefficient matrix determine the behavior of solutions and the general solution. Three-dimensional linear systems have three eigenvalues, so the list of possible qualitatively distinct phase spaces is longer than for planar systems. Since we must deal with three scalar equations rather than two, the arithmetic can quickly become much more involved. You might want to seek out software or a calculator capable of handling 3×3 matrices.

Linear Independence and the Linearity Principle

The general form of a linear system with three dependent variables is

$$\begin{aligned}
\frac{dx}{dt} &= a_{11}x + a_{12}y + a_{13}z \\
\frac{dy}{dt} &= a_{21}x + a_{22}y + a_{23}z \\
\frac{dz}{dt} &= a_{31}x + a_{32}y + a_{33}z,
\end{aligned}$$

where x, y, and z are the dependent variables and the coefficients a_{ij}, $(i, j = 1, 2, 3)$, are constants. We can write this system in matrix form as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

where $\mathbf{A}$ is the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and $\mathbf{Y}$ is the vector of dependent variables,

$$\mathbf{Y} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

To specify an initial condition for such a system, we must give three numbers, x_0, y_0, and z_0.

The Linearity Principle holds for linear systems in all dimensions, so if $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, then $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$ is also a solution for any constants k_1 and k_2.

Suppose $\mathbf{Y}_1(t)$, $\mathbf{Y}_2(t)$ and $\mathbf{Y}_3(t)$ are three solutions of the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$

If for any point (x_0, y_0, z_0) there exist constants k_1, k_2, and k_3 such that

$$k_1\mathbf{Y}_1(0) + k_2\mathbf{Y}_2(0) + k_3\mathbf{Y}_3(0) = (x_0, y_0, z_0),$$

then the general solution of the system is

$$\mathbf{Y}(t) = k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t) + k_3\mathbf{Y}_3(t).$$

In order for three solutions $\mathbf{Y}_1(t)$, $\mathbf{Y}_2(t)$, and $\mathbf{Y}_3(t)$ to give the general solution, the three vectors $\mathbf{Y}_1(0)$, $\mathbf{Y}_2(0)$, and $\mathbf{Y}_3(0)$ must point in "different directions"; that is, no one of them can be in the plane through the origin and the other two. In this case the vectors $\mathbf{Y}_1(0)$, $\mathbf{Y}_2(0)$, and $\mathbf{Y}_3(0)$ (and the corresponding solutions) are said to be *linearly independent*. We present an algebraic technique for checking linear independence in the exercises (see Exercises 2 and 3).

An example

Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \\ 0.4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We can check that the functions

$$\mathbf{Y}_1(t) = e^{0.2t} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\mathbf{Y}_2(t) = e^{-0.1t} \begin{pmatrix} -\cos\left(\sqrt{0.03}\,t\right) - \sqrt{3}\sin\left(\sqrt{0.03}\,t\right) \\ -2\cos\left(\sqrt{0.03}\,t\right) + 2\sqrt{3}\sin\left(\sqrt{0.03}\,t\right) \\ 4\cos\left(\sqrt{0.03}\,t\right) \end{pmatrix}$$

$$\mathbf{Y}_3(t) = e^{-0.1t} \begin{pmatrix} -\sin\left(\sqrt{0.03}\,t\right) + \sqrt{3}\cos\left(\sqrt{0.03}\,t\right) \\ -2\sin\left(\sqrt{0.03}\,t\right) - 2\sqrt{3}\cos\left(\sqrt{0.03}\,t\right) \\ 4\sin\left(\sqrt{0.03}\,t\right) \end{pmatrix}$$

are solutions by substituting them into the differential equation. For example,

$$\frac{d\mathbf{Y}_1}{dt} = e^{0.2t}\begin{pmatrix} 0.2 \\ 0.4 \\ 0.4 \end{pmatrix}$$

and

$$\mathbf{A}\mathbf{Y}_1(t) = \begin{pmatrix} 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \\ 0.4 & 0 & 0 \end{pmatrix} e^{0.2t}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = e^{0.2t}\begin{pmatrix} 0.2 \\ 0.4 \\ 0.4 \end{pmatrix},$$

so $\mathbf{Y}_1(t)$ is a solution. The other two functions can be checked similarly (see Exercise 1). We can sketch the solution curves that correspond to these solutions in the three-dimensional phase space (see Figure 3.56).

The initial conditions of these three solutions are $\mathbf{Y}_1(0) = (1, 2, 2)$, $\mathbf{Y}_2(0) = (-1, -2, 4)$, and $\mathbf{Y}_3(0) = (\sqrt{3}, -2\sqrt{3}, 0)$. These vectors are shown in Figure 3.57, where we can see that none of them is in the plane determined by the other two; hence, they are linearly independent.

For example, to find the solution $\mathbf{Y}(t)$ with initial position $\mathbf{Y}(0) = (2, 1, 3)$, we must solve

$$k_1\mathbf{Y}_1(0) + k_2\mathbf{Y}_2(0) + k_3\mathbf{Y}_3(0) = (2, 1, 3),$$

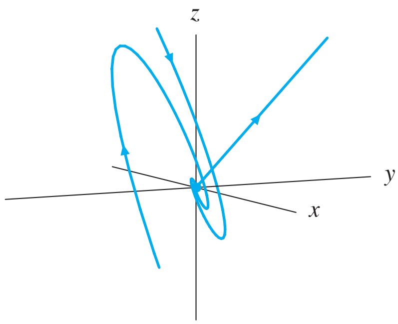

Figure 3.56
The solution curves of $\mathbf{Y}_1(t)$, $\mathbf{Y}_2(t)$, and $\mathbf{Y}_3(t)$.

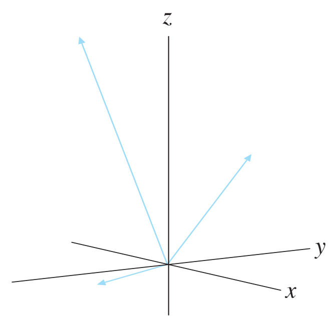

Figure 3.57
Vectors $\mathbf{Y}_1(0) = (1, 2, 2)$, $\mathbf{Y}_2(0) = (-1, -2, 4)$, and $\mathbf{Y}_3(0) = (\sqrt{3}, -2\sqrt{3}, 0)$ in xyz-space.

which is equivalent to

$$\begin{cases} k_1 - k_2 + \sqrt{3}k_3 = 2 \\ 2k_1 - 2k_2 - 2\sqrt{3}k_3 = 1 \\ 2k_1 + 4k_2 = 3. \end{cases}$$

We obtain $k_1 = 4/3$, $k_2 = 1/12$ and $k_3 = \sqrt{3}/4$, and the solution is

$$\mathbf{Y}(t) = \frac{4}{3}\mathbf{Y}_1(t) + \frac{1}{12}\mathbf{Y}_2(t) + \frac{\sqrt{3}}{4}\mathbf{Y}_3(t).$$

Eigenvalues and Eigenvectors

The method for finding solutions of systems with three dependent variables is the same as that for systems with two variables. We begin by finding eigenvalues and eigenvectors. Suppose we are given a linear system $d\mathbf{Y}/dt = \mathbf{AY}$, where $\mathbf{A}$ is a 3×3 matrix of coefficients and $\mathbf{Y} = (x, y, z)$. An *eigenvector* for the matrix $\mathbf{A}$ is a nonzero vector $\mathbf{V}$ such that

$$\mathbf{AV} = \lambda \mathbf{V},$$

where λ is the *eigenvalue* for $\mathbf{V}$. If $\mathbf{V}$ is an eigenvector for $\mathbf{A}$ with eigenvalue λ, then

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}$$

is a solution of the linear system.

The method for finding eigenvalues and eigenvectors for a 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is very similar to that for two-dimensional systems, only requiring more arithmetic. In particular, we need the formula for the determinant of a 3×3 matrix.

DEFINITION The *determinant* of the matrix $\mathbf{A}$ is

$$\det \mathbf{A} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$ ■

Using the 3×3 identity matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain the *characteristic polynomial* of $\mathbf{A}$ as

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix}.$$

As in the two-dimensional case, we have:

THEOREM The eigenvalues of a 3×3 matrix $\mathbf{A}$ are the roots of its characteristic polynomial. ■

To find the eigenvalues of a 3×3 matrix, we must find the roots of a cubic polynomial. This is not as easy as finding the roots of a quadratic. Although there is a "cubic equation" analogous to the quadratic equation for finding the roots of a cubic, it is quite complicated. (It is used by computer algebra packages to give exact values of roots of cubics.) However, in cases where the cubic does not easily factor, we frequently turn to numerical techniques such as Newton's method for finding roots.

To find the corresponding eigenvectors, we must solve a system of three linear equations with three unknowns. Luckily there are many examples of systems that illustrate the possible behaviors in three dimensions and for which the arithmetic is manageable.

A diagonal matrix

The simplest type of 3×3 matrix is a *diagonal* matrix—the only nonzero terms lie on the diagonal. For example, consider the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{A}$ is $(-3 - \lambda)(-1 - \lambda)(-2 - \lambda)$, which is simple because so many of the coefficients of $\mathbf{A}$ are zero. The eigenvalues are the roots of this polynomial, that is, the solutions of

$$(-3 - \lambda)(-1 - \lambda)(-2 - \lambda) = 0.$$

Thus the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = -1$, and $\lambda_3 = -2$.

Finding the corresponding eigenvectors is also not too hard. For $\lambda_1 = -3$, we must solve

$$\mathbf{AV}_1 = -3\mathbf{V}_1$$

for $\mathbf{V}_1 = (x_1, y_1, z_1)$. The product $\mathbf{AV}_1$ is

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} -3x_1 \\ -y_1 \\ -2z_1 \end{pmatrix},$$

and therefore we want to solve

$$\begin{pmatrix} -3x_1 \\ -y_1 \\ -2z_1 \end{pmatrix} = -3 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

for x_1, y_1, and z_1. Solutions of this system of three equations with three unknowns are $y_1 = z_1 = 0$ and x_1 may have any (nonzero) value. So, in particular, $(1, 0, 0)$ is an eigenvector for $\lambda_1 = -3$. Similarly, we find that $(0, 1, 0)$ and $(0, 0, 1)$ are eigenvectors for $\lambda_2 = -1$ and $\lambda_3 = -2$, respectively. Note that $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are linearly independent.

From these eigenvalues and eigenvectors we can construct solutions of the system

$$\mathbf{Y}_1(t) = e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{Y}_2(t) = e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-t} \\ 0 \end{pmatrix},$$

and

$$\mathbf{Y}_3(t) = e^{-2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ e^{-2t} \end{pmatrix}.$$

Because this system is diagonal, we could have gotten this far "by inspection." If we write the system in components

$$\begin{aligned} \frac{dx}{dt} &= -3x \\ \frac{dy}{dt} &= -y \\ \frac{dz}{dt} &= -2z, \end{aligned}$$

we see that dx/dt depends only on x, dy/dt depends only on y, and dz/dt depends only on z. In other words, the system completely decouples, and each coordinate can be dealt with independently. It is easy to solve these equations.

Now that we have three independent solutions, we can solve any initial-value problem for this system. For example, to find the solution $\mathbf{Y}(t)$ with $\mathbf{Y}(0) = (2, 1, 2)$, we must find constants k_1, k_2, and k_3 such that

$$(2, 1, 2) = k_1\mathbf{Y}_1(0) + k_2\mathbf{Y}_2(0) + k_3\mathbf{Y}_3(0).$$

So $k_1 = 2$, $k_2 = 1$, and $k_3 = 2$, and $\mathbf{Y}(t) = (2e^{-3t}, e^{-t}, 2e^{-2t})$ is the required solution.

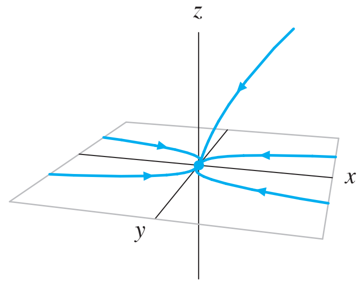

Figure 3.58
Phase space for $d\mathbf{Y}/dt = \mathbf{AY}$ for the diagonal matrix $\mathbf{A}$.

Figure 3.58 is a sketch of the phase space. Note that the coordinate axes are lines of eigenvectors, so they form straight-line solutions. Since all three of the eigenvalues are negative, solutions along all three of the axes tend toward the origin. Because every other solution can be made up as a linear combination of the solutions on the axes, all solutions must tend to the origin and it is natural to call the origin a sink.

Three-dimensional behavior

Before giving a classification of linear systems in three dimensions, we give an example whose qualitative behavior is different from that of any two-dimensional system.

Consider the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{BY} = \begin{pmatrix} 0.1 & -1 & 0 \\ 1 & 0.1 & 0 \\ 0 & 0 & -0.2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{B}$ is

$$((0.1-\lambda)(0.1-\lambda)+1)(-0.2-\lambda) = (\lambda^2 - 0.2\lambda + 1.01)(-0.2-\lambda),$$

so the eigenvalues are $\lambda_1 = -0.2$, $\lambda_2 = 0.1 + i$, and $\lambda_3 = 0.1 - i$. Corresponding to the real negative eigenvalue λ_1, we expect to see a line of solutions that approach the origin in the phase space. By analogy to the two-dimensional case, the complex eigenvalues with positive real part correspond to solutions that spiral away from the origin. This is a "spiral saddle," which is not possible in two dimensions.

We could find the eigenvectors associated with each eigenvalue as above and find the general solution. The eigenvectors for the complex eigenvalues are complex, and to find the real solutions, we would have to take real and imaginary parts, just as in two dimensions. However, we are lucky again, and this system also decouples into

$$\frac{dx}{dt} = 0.1x - y$$
$$\frac{dy}{dt} = x + 0.1y$$

and

$$\frac{dz}{dt} = -0.2z.$$

In the xy-plane, the eigenvalues are $0.1 \pm i$, so the origin is a spiral source. Along the z-axis, all solutions tend toward zero as time increases (see Figure 3.59).

Combining these pictures, we obtain a sketch of the three-dimensional phase space. Note that the z-coordinate of each solution decreases toward zero, while in the xy-plane solutions spiral away from the origin (see Figure 3.60).

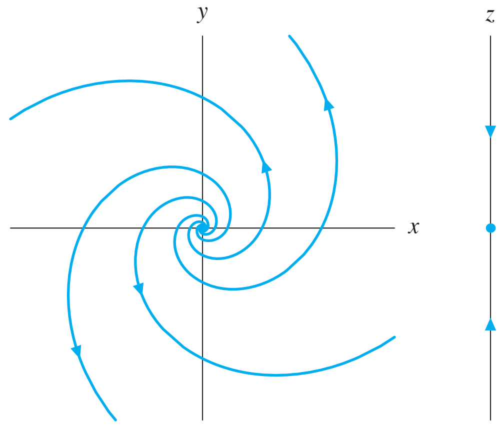

Figure 3.59
Phase plane for xy-system and phase line for z.

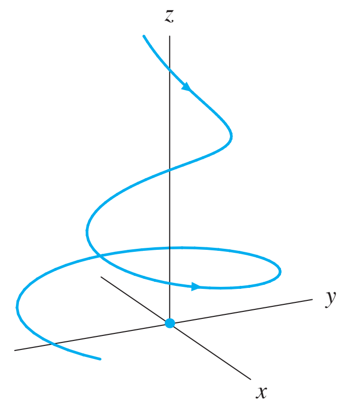

Figure 3.60
Phase space for $d\mathbf{Y}/dt = \mathbf{BY}$.

Classification of Three-Dimensional Linear Systems

Although there are more possible types of phase space pictures for three-dimensional linear systems than for two dimensions, the list is still finite. Just as for two dimensions, the nature of the system is determined by the eigenvalues. Real eigenvalues correspond to straight-line solutions that tend toward the origin if the eigenvalue is negative and away from the origin if the eigenvalue is positive. Complex eigenvalues correspond to spiraling. Negative real parts indicate spiraling toward the origin, whereas positive real parts indicate spiraling away from the origin.

Since the characteristic polynomial is a cubic, there are three eigenvalues (which might not all be distinct if there are repeated roots). It is always the case that at least one of the eigenvalues is real. The other two may be real or a complex conjugate pair (see exercises).

The most important types of three-dimensional linear systems can be divided into three categories: sinks, sources, and saddles. Examples of the other cases (which include systems with double eigenvalues and zero eigenvalues) are given in the exercises.

Sinks

We call the equilibrium point at the origin a sink if all solutions tend toward it as time increases. If all three eigenvalues are real and negative, then there are three straight lines of solutions, all of which tend toward the origin. Since every other solution is a linear combination of these solutions, all solutions tend to the origin as time increases (see Figure 3.58).

The other possibility for a sink is to have one real negative eigenvalue and two complex eigenvalues with negative real parts. This means that there is one straight line of solutions tending to the origin and a plane of solutions that spiral toward the origin. All other solutions exhibit both of these behaviors (see Figure 3.61).

Sources

There are two possibilities for sources as well. We can have either three real and positive eigenvalues or one real positive eigenvalue and a complex conjugate pair with positive real parts. An example of such a phase space is given in Figure 3.62. Note that this system looks just like the sink in Figure 3.61 except the directions of the arrows have been reversed, so solutions move away from the origin as time increases.

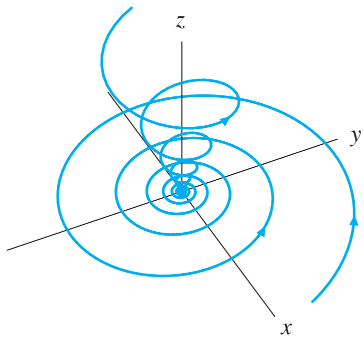

Figure 3.61
Example phase space for spiral sink.

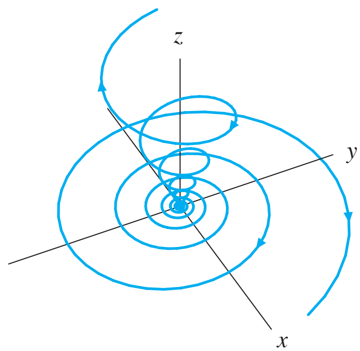

Figure 3.62
Example phase space for spiral source.

Saddles

The equilibrium point at the origin is a saddle if, as time increases to infinity, some solutions tend toward it while other solutions move away from it. This can occur in four different ways. If all the eigenvalues are real, then we could have one positive and two negative or two positive and one negative. In the first case, one positive and two negative, there is one straight line of solutions that tend away from the origin as time increases and a plane of solutions that tend toward the origin as time increases. In the other case, two positive and one negative, there is a plane of solutions that tend away from the origin as time increases and a line of solutions that tend toward the origin as time increases. In both cases, all other solutions will eventually move away from the origin as time increases or decreases (see Figure 3.63).

The other two cases occur if there is only one real eigenvalue and the other two are a complex conjugate pair. If the real eigenvalue is negative and the real parts of the complex eigenvalues are positive, then as time increases there is a straight line of solutions that tend toward the origin and a plane of solutions that tend away from it. All other solutions are a combination of these behaviors, so as time increases they spiral around the straight line of solutions in ever widening loops (see Figure 3.60). The other possibility is that the real eigenvalue is positive and the complex eigenvalues have negative real part. In this case there is a straight line of solutions that tend away from the origin as time increases and a plane of solutions that spiral toward the origin as time increases. Every other solution spirals around the straight line of solutions while moving away from the origin (see Figure 3.64).

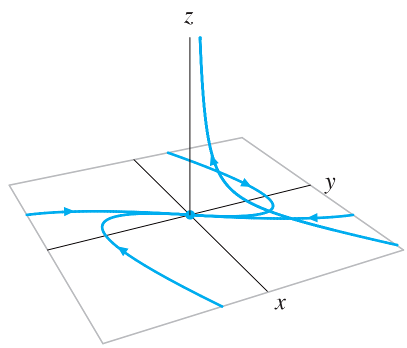

Figure 3.63
Example of a saddle with one positive and two negative eigenvalues.

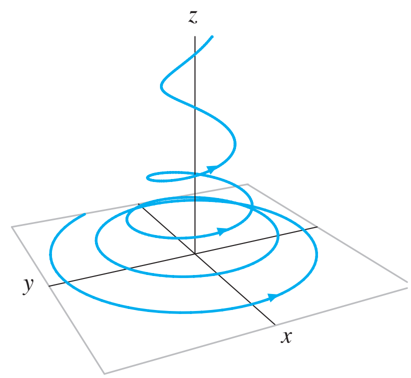

Figure 3.64
Example of a saddle with one real eigenvalue and a complex conjugate pair of eigenvalues.

An example revisited

We end this section by returning to the example that we used at the start of the section. All of the other examples in this section have been systems that decouple into systems of smaller dimension. Sadly, the general case is not so simple. This example doesn't look too complicated because the coefficient matrix has many zero entries. However, it does not immediately decouple into lower dimensional systems.

Consider the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \\ 0.4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The characteristic polynomial for $\mathbf{A}$ is $-\lambda^3 + 0.008$, so the eigenvalues are the solutions of

$$-\lambda^3 + 0.008 = 0.$$

That is, the eigenvalues are the cube roots of 0.008. Every number has three cube roots if we consider complex as well as real roots. The cube roots of 0.008 are $\lambda_1 = 0.2$, $\lambda_2 = 0.2e^{2\pi i/3}$, and $\lambda_3 = 0.2e^{-2\pi i/3}$. The last two eigenvalues may be written as $\lambda_2 = -0.1 + i\sqrt{0.03}$ and $\lambda_3 = -0.1 - i\sqrt{0.03}$.

This system is a saddle with one positive real eigenvalue and a complex conjugate pair of eigenvalues with negative real parts. Solutions spiral tightly around the line of eigenvectors associated to the eigenvalue $\lambda_1 = 0.2$. In order to sketch the phase space, we must find the eigenvectors for these eigenvalues.

For $\lambda_1 = 0.2$, the eigenvectors are solutions of

$$\mathbf{AV}_1 = 0.2\mathbf{V}_1,$$

which is written in coordinates as

$$\begin{cases} 0.1y_1 = 0.2x_1 \\ 0.2z_1 = 0.2y_1 \\ 0.4x_1 = 0.2z_1. \end{cases}$$

In particular $\mathbf{V}_1 = (1/2, 1, 1)$ is one such eigenvector. The vector $\mathbf{V}_1$ can be used to determine an entire line of eigenvectors in space.

To find the plane of solutions that spiral toward the origin, we must find the eigenvectors for $\lambda_2 = -0.1 + i\sqrt{0.03}$. That is, we must solve

$$\mathbf{AV}_2 = \left(-0.1 + i\sqrt{0.03}\right)\mathbf{V}_2$$

for $\mathbf{V}_2$. In other words,

$$\begin{aligned} y_2 &= \left(-1 + i\sqrt{3}\right)x_2 \\ 2z_2 &= \left(-1 + i\sqrt{3}\right)y_2 \\ 4x_2 &= \left(-1 + i\sqrt{3}\right)z_2. \end{aligned}$$

One eigenvector associated to λ_2 is $\mathbf{V}_2 = (-1+i\sqrt{3}, -2-i2\sqrt{3}, 4)$. The corresponding solution to the system is

$$\mathbf{Y}_2(t) = e^{\left(-0.1+i\sqrt{0.03}\right)t}\left(-1 + i\sqrt{3}, -2 - i2\sqrt{3}, 4\right).$$

We can convert this into two real-valued solutions by taking real and imaginary parts. Since our goal is to find the plane on which solutions spiral, we need only look at the initial point $\mathbf{Y}_2(0) = (-1 + i\sqrt{3}, -2 - i2\sqrt{3}, 4)$. The initial points of the real and imaginary parts are $(-1, -2, 4)$ and $(\sqrt{3}, -2\sqrt{3}, 0)$, respectively. The plane on which solutions spiral toward the origin is the plane made up of all linear combinations of these two vectors. We can use this information to give a fairly accurate sketch of the phase space of this system (see Figure 3.56). We also sketch the graphs of the coordinate functions for one solution (see Figures 3.65 and 3.66). Note that for the example

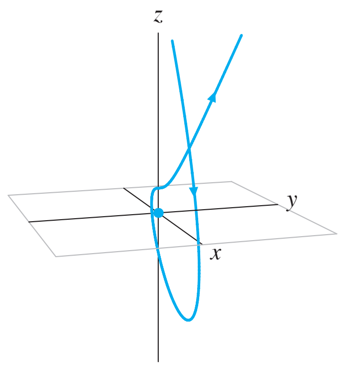

Figure 3.65
Phase space for system $d\mathbf{Y}/dt = \mathbf{AY}$.

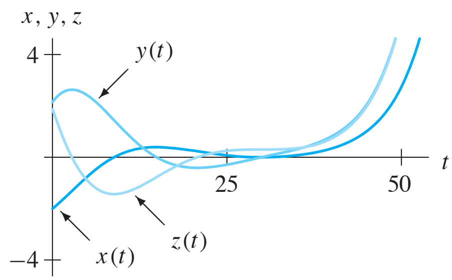

Figure 3.66
Graphs of $x(t)$, $y(t)$ and $z(t)$ for the indicated solution in Figure 3.65.

solution shown, all three coordinates tend to infinity as t increases because the eigenvector for the eigenvalue λ_1 has nonzero components for all three variables.

Three linearly independent solutions of this system are given in the first example of this section (see page 361). We can see from this example that linear systems in three dimensions can be quite complicated (even when many of the coefficients are zero). However, the qualitative behavior is still determined by the eigenvalues, so it is possible to classify these systems without completely solving them.

EXERCISES FOR SECTION 3.8

1. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \\ 0.4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Check that the functions

$$\mathbf{Y}_2(t) = e^{-0.1t} \begin{pmatrix} -\cos\left(\sqrt{0.03}\,t\right) - \sqrt{3}\sin\left(\sqrt{0.03}\,t\right) \\ -2\cos\left(\sqrt{0.03}\,t\right) + 2\sqrt{3}\sin\left(\sqrt{0.03}\,t\right) \\ 4\cos\left(\sqrt{0.03}\,t\right) \end{pmatrix}$$

and

$$\mathbf{Y}_3(t) = e^{-0.1t} \begin{pmatrix} -\sin\left(\sqrt{0.03}\,t\right) + \sqrt{3}\cos\left(\sqrt{0.03}\,t\right) \\ -2\sin\left(\sqrt{0.03}\,t\right) - 2\sqrt{3}\cos\left(\sqrt{0.03}\,t\right) \\ 4\sin\left(\sqrt{0.03}\,t\right) \end{pmatrix}$$

are solutions to the system.

2. If a vector $\mathbf{Y}_3$ lies in the plane determined by the two vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$, then we can write $\mathbf{Y}_3$ as a linear combination of $\mathbf{Y}_1$ and $\mathbf{Y}_2$. That is,

$$\mathbf{Y}_3 = k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2$$

for some constants k_1 and k_2. But then

$$k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2 - \mathbf{Y}_3 = (0, 0, 0).$$

Show that if

$$k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2 + k_3\mathbf{Y}_3 = (0, 0, 0),$$

with not all of k_1, k_2, and $k_3 = 0$, then the vectors are not linearly independent. [*Hint*: Start by assuming that $k_3 \neq 0$ and show that $\mathbf{Y}_3$ is in the plane determined by $\mathbf{Y}_1$ and $\mathbf{Y}_2$. Then treat the other cases.] Note that this computation leads to the theorem that three vectors $\mathbf{Y}_1$, $\mathbf{Y}_2$, and $\mathbf{Y}_3$ are linearly independent if and only if the only solution of

$$k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2 + k_3\mathbf{Y}_3 = (0, 0, 0)$$

is $k_1 = k_2 = k_3 = 0$.

3. Using the technique of Exercise 2, determine whether or not the following sets of three vectors are linearly independent.

(a) $(1, 2, 1), (1, 3, 1), (1, 4, 1)$

(b) $(2, 0, -1), (3, 2, 2), (1, -2, -3)$

(c) $(1, 2, 0), (0, 1, 2), (2, 0, 1)$

(d) $(-3, \pi, 1), (0, 1, 0), (-2, -2, -2)$

In Exercises 4–7, consider the linear system $d\mathbf{Y}/dt = \mathbf{AY}$ with the coefficient matrix $\mathbf{A}$ specified. Each of these systems decouples into a two-dimensional system and a one-dimensional system. For each exercise,

(a) compute the eigenvalues,

(b) determine how the system decouples,

(c) sketch the two-dimensional phase plane and one-dimensional phase line for the decoupled systems, and

(d) give a rough sketch of the phase portrait of the system.

4. $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

5. $\mathbf{A} = \begin{pmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

6. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$

7. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Exercises 8–9 consider the properties of the cubic polynomial

$$p(\lambda) = \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta,$$

where α, β, γ, and δ are real numbers.

8. **(a)** Show that, if α is positive, then the limit of $p(\lambda)$ as $\lambda \to \infty$ is ∞ and the limit of $p(\lambda)$ as $\lambda \to -\infty$ is $-\infty$.

(b) Show that, if α is negative, then the limit of $p(\lambda)$ as $\lambda \to \infty$ is $-\infty$ and the limit of $p(\lambda)$ as $\lambda \to -\infty$ is ∞.

(c) Using the above, show that $p(\lambda)$ must have at least one real root (that is, at least one real number λ_0 such that $p(\lambda_0) = 0$). [*Hint*: Look at the graph of $p(\lambda)$.]

9. Suppose $a + ib$ is a root of $p(\lambda)$ (so $p(a + ib) = 0$). Show that $a - ib$ is also a root. [*Hint*: Remember that a complex number is zero if and only if both its real and imaginary parts are zero. Then compute $p(a + ib)$ and $p(a - ib)$.]

In Exercises 10–13, consider the linear system $d\mathbf{Y}/dt = \mathbf{BY}$ with the coefficient matrix $\mathbf{B}$ specified. These systems do not fit into the classification of the most common types of systems given in the text. However, the equations for dx/dt and dy/dt decouple from dz/dt. For each of these systems,

(a) compute the eigenvalues,

(b) sketch the xy-phase plane and the z-phase line, and

(c) give a rough sketch of the phase portrait of the system.

10. $\mathbf{B} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

11. $\mathbf{B} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

12. $\mathbf{B} = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

13. $\mathbf{B} = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

In Exercises 14–15, consider the linear system $d\mathbf{Y}/dt = \mathbf{CY}$. These systems do not fit into the classification of the most common types of systems given in the text, and they do not decouple into lower-dimensional systems. For each system,

(a) compute the eigenvalues,

(b) compute the eigenvectors, and

(c) sketch (as best you can) the phase portrait of the system. [*Hint*: Use the eigenvalues and eigenvectors and also vectors in the vector field.]

14. $\mathbf{C} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$

15. $\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

16. For the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}:$$

(a) Show that $\mathbf{V}_1 = (1, 1, 1)$ is an eigenvector of the coefficient matrix by computing $\mathbf{AV}_1$. What is the eigenvalue for this eigenvector?

(b) Find the other two eigenvalues for the matrix $\mathbf{A}$.

(c) Classify the system (source, sink, ...).

(d) Sketch (as best you can) the phase portrait. [*Hint*: Use the other eigenvalues and find the other eigenvectors.]

17. For the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} -4 & 3 & 0 \\ 0 & -1 & 1 \\ 5 & -5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}:$$

(a) Show that $\mathbf{V}_1 = (1, 1, 0)$ is an eigenvector of the coefficient matrix by computing $\mathbf{AV}_1$. What is the eigenvalue for this eigenvector?

(b) Find the other two eigenvalues for the matrix $\mathbf{A}$.

(c) Classify the system (source, sink, ...).

(d) Sketch (as best you can) the phase portrait. [*Hint*: Use the other eigenvalues and find the other eigenvectors.]

18. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{BY} = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(This system is related to the Lorenz system studied in Section 2.8, and we will use the results obtained in this exercise when we return to the Lorenz equations in Section 5.5.)

(a) Find the characteristic polynomial and the eigenvalues.

(b) Find the eigenvectors.

(c) Sketch the phase portrait (as best you can).

(d) Comment on how the fact that the system "decouples" helps in the computations and in sketching the phase space.