Exercise 5.10 (Synopsis): \((A \subseteq B) \text{ iff } (A \cup B) = B\).

Solution:

(* State type of proof*)

1. This is a 2-way proof. I’ll first show \((A \subseteq B) \Rightarrow (A \cup B) = B\) [in a direct proof].

(* State your hypotheses*)

2. Let \(A\) and \(B\) be sets such that \(A \subseteq B\).

(* Signpost of strategy*)

3. I’ll show \((A \cup B) = B\) by proving subsets in both directions.
   - I’ll first show \((A \cup B) \subseteq B\).

(* Start your element argument*)

4. Let \(m \in (A \cup B)\).
   - * Use logic, together with hypotheses, to trace your element's properties*

5. Then \(m \in A\) or \(m \in B\). [by definition of union]

6. By hypothesis, \(m \in A \Rightarrow m \in B\). Therefore, line (5) implies \(m \in B\).

7. From lines (4) & (6) we have: \(m \in (A \cup B) \Rightarrow m \in B\).
   - It follows that \((A \cup B) \subseteq B\).

(* Signpost next move*)

8. Next I’ll show \(B \subseteq (A \cup B)\). Let \(n \in B\).

9. Then \(n \in (B \cup A)\) for any set \(A\). [by definition of union]
   - Thus, \(n \in B \Rightarrow n \in (A \cup B)\) and we have \(B \subseteq (A \cup B)\).

(* State interim conclusion*)

10. Lines (2), (7) and (9) show that \((A \subseteq B) \Rightarrow (A \cup B) = B\).

(* Signpost next stage*)

11. Next I’ll prove the converse: \(((A \cup B) = B) \Rightarrow (A \subseteq B)\).

(* State hypotheses*)

12. Let \(A\) and \(B\) be sets such that \((A \cup B) = B\).

(* Signpost*)

13. To show \(A \subseteq B\), consider any \(x \in A\).

14. Then \(x \in (A \cup B)\). [by definition of union]

15. This implies \(x \in B\) [since, by hypothesis, \((A \cup B) = B\)]

16. Lines (13) and (15) show \(A \subseteq B\).

17. From lines (12) and (16) we conclude: \(((A \cup B) = B) \Rightarrow (A \subseteq B)\).

(* Conclusion*)

18. Lines (10) and (17) together prove the implication in both directions, completing the proof.
Exercise 5.18 (synopsis): Prove that the empty set is unique (i.e., if $A$ and $B$ are empty sets then $A = B$).

Solution:

(1) I will use a proof by contradiction.

(2) Let $A$ and $B$ be empty sets.

(3) By way of contradiction, suppose $A \neq B$.

(4) By negating the definition of set equality, this implies: $A \not\subseteq B$ or $B \not\subseteq A$.

(5) Let us first consider $A \not\subseteq B$.

(6) Since $A \not\subseteq B$, there exists $p \in A$ such that $p \notin B$. [by negating definition of subset]

(7) But $p \in A$ is a contradiction, because $A$ is empty! [by definition of empty set]

(8) By a similar argument $B \not\subseteq A$ also results in a contradiction.

(9) Lines (7) and (8) show that $A \subseteq B$ and $B \subseteq A$.

(10) By definition of set equality, $A = B$.

(11) From lines (2) and (10) we can conclude that if $A$ and $B$ are empty sets then $A = B$. It follows that the empty set is unique.