Exercise 2.3.4: For all sets $A$, $B$, and $C$, if $(A \cap C) \subseteq (B \cap C)$ and $(A \cup C) \subseteq (B \cup C)$, then $A \subseteq B$.

Solution: We will use a proof by contradiction.

\{ * State your hypotheses \}
1) Let $A$, $B$, and $C$ be sets such that $(A \cap C) \subseteq (B \cap C)$ and $(A \cup C) \subseteq (B \cup C)$.

\{ * State your contradictory supposition \}
2) By way of contradiction, suppose $A \nsubseteq B$.

\{ * Pick a suitable element to start your element argument \}
3) Since $A \nsubseteq B$, there exists $x \in A$ such that $x \notin B$. \[ by \ negating \ definition \ of \ subset \]

\{ * Use logic, together with hypotheses, to trace your element’s properties \}
4) Because $x \in A$, we know $x \in A \cup C$. \[ by \ definition \ of \ union \]
5) By our hypotheses, this implies $x \in B \cup C$. \[ because \ (A \cup C) \subseteq (B \cup C) \]
6) It follows that $x \in B$ or $x \in C$. \[ by \ definition \ of \ union \]

7) We consider each of the two possibilities

\( 7.1 \) $x \in B$ $\Rightarrow$ contradiction with line (3): $x \in B$ and $x \notin B$.

\( 7.2 \) $x \in C$ $\Rightarrow$ $x \in A$ and $x \in C$ $\Rightarrow$ $x \in A \cap C$. \[ line \ (3) \ & definition \ of \ intersection \]

But, $x \in A \cap C$ $\Rightarrow$ $x \in B \cap C$. \[ because \ by \ hypotheses \ A \cap C \subseteq B \cap C \]

Thus, we can conclude $x \in B$ and $x \in C$. \[ by \ definition \ of \ intersection \]

This contradicts line (3) because: $x \in B$ and $x \notin B$.

\{ * Tie your results together in a concluding statement \}
8) Therefore, we have shown $A \nsubseteq B$ leads to a contradiction in every case. It follows that if $(A \cap C) \subseteq (B \cap C)$ and $(A \cup C) \subseteq (B \cup C)$, then $A \subseteq B$.

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Exercise 2.3.6: Let $A$, $B$, and $X$ be sets. If $(A \cup B) \subseteq X$ and $(X - B) \subseteq (X - A)$, then $A \subseteq B$.

Solution: We will use a proof by contradiction.

\{ * State your hypotheses \}
1) Let $A$, $B$, and $X$ be sets such that $(A \cup B) \subseteq X$ and $(X - B) \subseteq (X - A)$.

\{ * State your contradictory supposition \}
2) By way of contradiction, suppose $A \nsubseteq B$.

\{ * Pick a suitable element to start your element argument \}
3) Since $A \nsubseteq B$, there exists $p \in A$ such that $p \notin B$. \[ by \ negating \ definition \ of \ subset \]

\{ * Use logic, together with hypotheses, to trace your element’s properties \}
4) Because $p \in A$, we know $p \in A \cup B$. \[ by \ definition \ of \ union \]
5) By our hypotheses, this implies $p \in X$. \[ because \ (A \cup B) \subseteq X \]
6) Lines (5) and (3) imply $p \in X$ and $p \notin B$. Thus $p \in X - B$. \[ definition \ of \ complement \]
7) Since $p \in X - B$, we conclude $p \in X - A$. \[ since \ (X - B) \subseteq (X - A) \ by \ hypothesis \]
8) $p \in X - A$ $\Rightarrow$ $p \in X$ and $p \notin A$. \[ definition \ of \ complement \]
9) This contradicts line (3): $p \in A$ and $p \notin A$.

\{ * Tie your results together in a concluding statement \}
10) Therefore, we have shown $A \nsubseteq B$ leads to a contradiction. It follows that if $(A \cup B) \subseteq X$ and $(X - B) \subseteq (X - A)$, then $A \subseteq B$.
Exercise 2.4.2: Let $A$, $B$, and $X$ be sets. Then $(X - A) \cup (X - B) \subseteq X - (A \cap B)$.

Solution: We will use a direct proof.

\{* State your hypotheses\}
(1) Let $A$, $B$, and $X$ be sets.

\{* State what you propose to prove\}
(2) I will show that $(X - A) \cup (X - B) \subseteq X - (A \cap B)$.

\{* Pick a suitable element to start your element argument\}
(3) Let $q$ be any element of the set $(X - A) \cup (X - B)$.

This means we’re assuming it is non-empty – otherwise there is nothing to prove!

\{* Use logic, together with hypotheses, to trace your element’s properties \}
(4) $q \in (X - A) \cup (X - B) \Rightarrow q \in (X - A)$ or $q \in (X - B)$. [by definition of union]

(5) We consider each of the two possibilities separately

\( (\text{5.1}) \quad q \in (X - A). \)

\( (\text{5.1.1}) \quad q \in (X - A) \Rightarrow q \in X \text{ and } q \notin A. \) [definition of complement]

\( (\text{5.1.2}) \quad q \notin A \Rightarrow q \notin (A \cap B). \) [by negating definition of intersection]

\( (\text{5.1.3}) \quad \text{From (5.1.1) and (5.1.2): } q \in X \text{ and } q \notin (A \cap B). \)

Therefore, $q \in X - (A \cap B)$. [by definition of complement]

\( (\text{5.2}) \quad q \in (X - B). \) By similar argument to (5.1) we have

\( (\text{5.2.1}) \quad q \in X \text{ and } q \notin B \Rightarrow q \notin (A \cap B) \Rightarrow q \in X - (A \cap B). \)

\{* Tie your results together in a concluding statement\}
(6) Therefore, we’ve shown $q \in (X - A) \cup (X - B) \Rightarrow q \in X - (A \cap B)$ for every case.

It follows from the definition of subset that $(X - A) \cup (X - B) \subseteq X - (A \cap B)$.

Exercise 2.6.3: For all sets $A$, $B$, $C$ and $D$, if $A \subseteq C$ and $B \subseteq D$ then $(A \cup B) \subseteq (C \cup D)$.

Solution: We will use a direct proof.

\{* State your hypotheses\}
(1) Let $A$, $B$, $C$, and $D$ be sets such that $A \subseteq C$ and $B \subseteq D$.

\{* State what you propose to prove\}
(2) I will show that $(A \cup B) \subseteq (C \cup D)$.

\{* Pick a suitable element to start your element argument\}
(3) Let $s$ be any element of the set $A \cup B$ (assumed non-empty – otherwise proof is vacuous).

\{* Use logic, together with hypotheses, to trace your element’s properties \}
(4) $s \in (A \cup B) \Rightarrow s \in A$ or $s \in B$. [by definition of union]

(5) We consider each of the two possibilities separately

\( (\text{5.1}) \quad \text{Suppose } s \in A. \)

\( (\text{5.1.1}) \quad \text{Then } s \in C. \quad \text{[by hypothesis } A \subseteq C, \text{ and by definition of subset]} \)

\( (\text{5.1.2}) \quad \text{This implies } s \in (C \cup D). \quad \text{[by definition of union]} \)

\( (\text{5.2}) \quad \text{Suppose } s \in B. \) By similar argument to (5.1) we have

\( (\text{5.2.1}) \quad s \in D \text{ because } B \subseteq D, \text{ which implies } s \in (D \cup C) \text{ by definition of union.} \)

\{* Tie your results together in a concluding statement\}
(6) Therefore, we’ve shown $s \in (A \cup B) \Rightarrow s \in (C \cup D)$ for every case.

It follows from the definition of subset that $(A \cup B) \subseteq (C \cup D)$.
Exercise 2.7.3: Let $A$, $B$, and $X$ be sets. If $(A \cup B) \subseteq X$ then $(A - B) = (X - B) - (X - A)$.

Solution: We will use a direct proof.

{ * State your hypotheses }
1. Let $A$, $B$, and $X$ be sets such that $(A \cup B) \subseteq X$.

{ * State what you propose to prove }
2. To prove $(A - B) = (X - B) - (X - A)$, I must show a subset relationship in both directions. I will first show that $(A - B) \subseteq (X - B) - (X - A)$.

{ * Pick a suitable element to start your element argument }
3. Let $m$ be any element of the set $A - B$.

{ * Use logic, together with hypotheses, to trace your element’s properties }
4. This implies $m \in A$ and $m \notin B$. \[ \text{[by definition of complement]} \]
5. Thus, $m \in (A \cup B)$. \[ \text{[by definition of union, because } m \in A] \]
6. $\Rightarrow m \in X$. \[ \text{[because } (A \cup B) \subseteq X \text{ by hypothesis]} \]
7. Lines (6) and (4) imply $m \in (X - B)$. \[ \text{[because } m \in X \text{ and } m \notin B] \]
8. Lines (6) and (4) also show $m \notin (X - A)$. \[ \text{[because } m \in A \Rightarrow m \notin (X - A)] \]
9. From (7) and (8) we have $m \in [(X - B) - (X - A)]$. \[ \text{[by definition of complement]} \]

{ * Tie your results together in a concluding statement }
10. Therefore, we’ve shown via (3) and (9): $m \in (A - B) \Rightarrow m \in [(X - B) - (X - A)]$.

It follows from the definition of subset that $(A - B) \subseteq (X - B) - (X - A)$.

{ * Prepare reader for proof in other direction }
11. Next, I will show $(X - B) - (X - A) \subseteq (A - B)$.

{ * Pick a suitable element to start your element argument }
12. Let $p$ be any element of the set $(X - B) - (X - A)$.

{ * Use logic, together with hypotheses, to trace your element’s properties }
13. Then $p \in (X - B)$ and $p \notin (X - A)$. \[ \text{[definition of complement]} \]
14. $p \in (X - B) \Rightarrow p \in X$ and $p \notin B$. \[ \text{[definition of complement]} \]
15. Also, $p \notin (X - A) \Rightarrow p \notin X$ or $p \in A$. \[ \text{[by negation of complement definition]} \]
16. The only way for lines (14) and (15) to simultaneously hold is: $p \notin B$ and $p \in A$.

It follows that $p \in (A - B)$. \[ \text{[definition of complement]} \]

{ * Tie together final result }
17. We have shown via (12) and (16) that $p \in [(X - B) - (X - A)] \Rightarrow p \in (A - B)$.

Therefore, $(X - B) - (X - A) \subseteq (A - B)$.

18. Lines (10) and (17) taken together show that $(A - B) = (X - B) - (X - A)$. \[ \text{[by definition of set equality]} \]