Mathematical Modelling with Case Studies

A differential equation approach using Maple

Belinda Barnes

School of Mathematical Sciences
Australian National University
Canberra

and

Glenn R. Fulford

School of Mathematics and Statistics
The University of New South Wales (ADFA)
Canberra

London and New York

7.1 Introduction

motivation

From the small sample of examples we have examined in the previous chapters, we have seen that systems of equations can result in many different types of behaviour. Depending on the initial conditions or the chosen parameter values, the outcome may be stable or unstable, cyclic or divergent.

overview

In this chapter we develop some powerful theory which allows us to predict the dynamical behaviour of a system. In the first case, we consider only linear systems of equations. However, since many of the interacting population models we have met are nonlinear (and in fact most natural systems are nonlinear), we then show how this theory for the linear case may be extended to nonlinear systems as well. In the final section we apply the process to the nonlinear models of population interactions, which we developed in Chapters 5 and 6, and shall extend in Chapter 8.

7.2 Linear theory

observed dynamics

So far, in our phase-plane analysis of systems of equations, we have encountered a variety of behaviours of trajectories near equilibrium points. For example we saw trajectories approaching some of these points, trajectories being repelled by others, as well as spiralling trajectories and closed loops.

overview

In the following theory we develop techniques which allow us to predict, for each equilibrium point, the behaviour of the trajectories close to that point. From this we can establish a complete picture of the system phase-plane. Initially, we consider the linear case (a pair of coupled linear equations in two unknowns) and then show how this can be extended to the nonlinear case. (Whilst we restrict our analysis to two equations in two unknowns, the theory is applicable to larger systems with many unknowns.)

The general linear system

general linear system

We start by considering the following general form of a pair of coupled linear equations:

$$X' = a_1X + b_1Y$$
$$Y' = a_2X + b_2Y$$

where differentiation is with respect to time t (i.e. $X' = dX/dt$, $Y' = dY/dt$) and a_1, a_2, b_1 and b_2 are constant.

We denote an equilibrium point (critical point or steady-state) for the system by (x_e, y_e). Thus $a_1x_e + b_1y_e = 0$ and $a_2x_e + b_2y_e = 0$.

general equilibrium point

Linear algebra notation

The system above can be written in terms of matrices and vectors in the following way. Let

general system in vector notation

$$\mathbf{x} = \begin{bmatrix} X \\ Y \end{bmatrix} \quad \text{and} \quad \mathbf{x}' = \begin{bmatrix} X' \\ Y' \end{bmatrix}$$

where x is a vector. Let

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

where A is a matrix. Then the above system of two equations can be written as

$$\mathbf{x}' = A\mathbf{x}.$$

This means that

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a_1X + b_1Y \\ a_2X + b_2Y \end{bmatrix} \qquad (1)$$

using normal matrix multiplication.

What needs to be understood in general, is the effect of multiplying a vector by a matrix A. We take an example to establish this.

effect of multiplying by a matrix

Example 1: *Carry out the multiplication $A\mathbf{x}_1$ and $A\mathbf{x}_2$ where*

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Solution:

$$A\mathbf{x}_1 = \begin{bmatrix} -3-2 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

and

$$A\mathbf{x}_2 = \begin{bmatrix} 6-2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

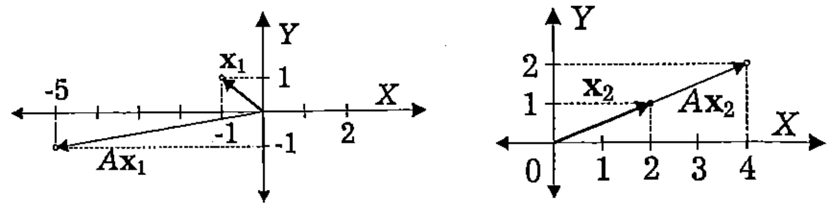

Figure 7.2.1: The effect of multiplying vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ by a matrix A as defined in Example 1.

interpretation

These results are illustrated in Figure 7.2.1. So multiplication by a matrix A maps a vector onto another vector. In the case of $\mathbf{x}_2$ we have that

$$A\mathbf{x}_2 = 2\mathbf{x}_2,$$

so that the effect of multiplying by A is the same as multiplying by a scalar or number (which is 2 in this case). We use this notion of the 'equivalence' of multiplication by a matrix and a scalar (number) in the process of finding eigenvalues and eigenvectors: the latter will be the vectors for which there is a nontrivial solution to the equation, and the former will be the associated scalars. These values turn out to be essential in predicting the behaviour of trajectories in the phase-plane associated with the system. For further details of matrix algebra see Appendix B.1.

Outline of method to solve equations

method outline

In order to solve a general system of equations, as in (1), we can transform the equations onto a different system of axes. In choosing the new axes carefully, the equations transform to simple differential equations of exponential growth or decay, and thus are simple to solve. Finally, these solutions can be transformed back to the original system of axes to provide a solution in the required form.

The eigenvalues and eigenvectors

the process

In order to transform a general pair of differential equations, which may be hard to solve, into a system which is easy to solve, we find and use the eigenvalues and eigenvectors. (These are described fully in Appendix B.1.) Essentially, *eigenvectors* $\mathbf{x}$ are the non-zero solutions of the matrix equation

$$A\mathbf{x} = \lambda\mathbf{x}$$

where the *eigenvalues* λ are the values for which these non-zero solutions exist.

Rewriting the equation as $A\mathbf{x} - \lambda\mathbf{x} = 0$ and then expanding using matrix multiplication we get

characteristic equation

$$A\mathbf{x} - \lambda\mathbf{x} = (A - \lambda I)\mathbf{x} \begin{bmatrix} a_1 - \lambda & b_1 \\ a_2 & b_2 - \lambda \end{bmatrix} \mathbf{x},$$

where I is the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From the theory of linear algebra (see Appendix B.1) it follows that for non-zero solutions of this equation to exist, the determinant of the expression must be zero, and thus we get the *characteristic equation* of matrix A

$$|A - \lambda I| = \lambda^2 - \lambda(a_1 + b_2) + (a_1 b_2 - a_2 b_1).$$

This equation is central to the theory which is developed here. Note that the coefficient of λ is the sum of the diagonal elements of the matrix A, namely the *trace* of A, and also that the last term is the *determinant* of matrix A. We use these values extensively when applying the theory which we now develop.

getting the eigenvalues and eigenvectors

We can establish the eigenvalues as the solutions to this characteristic equation. (We could also then solve for the associated pair of eigenvectors from the vector equation, but do not need these values in the applications.) From here the trajectory behaviour can be determined since, as will become apparent, it is dependent entirely on the eigenvalues.

Establishing the trajectory behaviour

the linear system

We return to the general pair of linear first-order equations

$$X' = a_1 X + b_1 Y,$$
$$Y' = a_2 X + b_2 Y,$$

which has an equilibrium point at the origin, $(x_e, y_e) = (0, 0)$. In vector notation then

$$\mathbf{x}' = A\mathbf{x}.$$

the matrix of eigenvectors

Suppose we have found the eigenvalues, λ_1 and λ_2, as well as the associated eigenvectors for A, namely

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

We define U to be the matrix whose columns are the eigenvectors, thus

$$U = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}.$$

the diagonal matrix of eigenvalues

From the definition of eigenvectors and eigenvalues we have

$$A\mathbf{u} = \lambda_1 \mathbf{u} \quad \text{and} \quad A\mathbf{v} = \lambda_2 \mathbf{v},$$

which implies that

$$A\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{u} & \lambda_2 \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{or} \quad AU = UD$$

with

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Assuming that U is invertible, we can write

$$U^{-1}AU = D. \tag{2}$$

We use this equation below.

x as a linear combination of the eigenvectors

First we express $\mathbf{x}$ as a linear combination of the eigenvectors and, assuming this is possible, we have

$$\mathbf{x} = z_1 \mathbf{u} + z_2 \mathbf{v}.$$

Letting

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad \text{then} \quad \mathbf{x} = U\mathbf{z}.$$

Since X and Y are functions of time, and the eigenvectors are not (since A is not a function of time), then z_1 and z_2 must also be functions of time. We now establish two expressions for $\mathbf{x}'$:

$$\mathbf{x} = U\mathbf{z} \qquad \text{so} \quad \mathbf{x}' = U\mathbf{z}',$$

and also

$$\mathbf{x}' = A\mathbf{x} \qquad \text{so} \quad \mathbf{x}' = AU\mathbf{z}.$$

Equating these two expressions for $\mathbf{x}'$ and then using (2) gives

$$U\mathbf{z}' = AU\mathbf{z},$$

and then

$$\begin{aligned} \mathbf{z}' &= U^{-1}AU\mathbf{z} \\ &= D\mathbf{z}. \end{aligned}$$

behaviour of trajectories

We are now in a position to solve the differential equations easily. Expanding $\mathbf{z}' = D\mathbf{z}$

$$\begin{aligned} z_1' &= \lambda_1 z_1, \\ z_2' &= \lambda_2 z_2, \end{aligned}$$

we obtain two equations which are easy to solve. They are the equations for exponential growth and decay with which, by now, we are familiar. We have as solutions $z_1 = k_1 e^{\lambda_1 t}$ and $z_2 = k_2 e^{\lambda_2 t}$ where k_1 and k_2 are arbitrary constants.

solutions for X and Y

Using these we can find solutions for X and Y by retracing our steps through this process and carrying the solutions with us. We have

$$\begin{aligned} \mathbf{x} &= k_1 e^{\lambda_1 t}\mathbf{u} + k_2 e^{\lambda_2 t}\mathbf{v} \\ &= e^{\lambda_1 t}\hat{\mathbf{u}} + e^{\lambda_2 t}\hat{\mathbf{v}}, \end{aligned}$$

where $\hat{\mathbf{u}} = k_1\mathbf{u}$ and $\hat{\mathbf{v}} = k_2\mathbf{v}$ are two eigenvectors (as any scalar multiple of an eigenvector is again an eigenvector) and so

$$\begin{aligned} X &= e^{\lambda_1 t}\hat{u_1} + e^{\lambda_2 t}\hat{v_1}, \\ Y &= e^{\lambda_1 t}\hat{u_2} + e^{\lambda_2 t}\hat{v_2}. \end{aligned}$$

Geometric interpretation

process summary

What have we done?

- We start with a set of axes, X and Y.
- We find eigenvectors $\mathbf{u}$ and $\mathbf{v}$ which give us the directions of a new system of axes z_1 and z_2 along which the effect of multiplying by A is the same as multiplying by a scalar λ. (It is clear from the diagram that any scalar multiple of the eigenvectors will suffice as an eigenvector.)
- We consider the plane described by the axes z_1 and z_2 and examine the behaviour of the solutions (which are now easy to find) $z_1 = k_1 e^{\lambda_1 t}$ and $z_2 = k_2 e^{\lambda_2 t}$ in this phase-plane. How the trajectories behave depends entirely on λ_1 and λ_2, the eigenvalues. Suppose they are as illustrated in the left-hand diagram of Figure 7.2.2.

- Next we return to the (X, Y) phase-plane with the transformation of this solution, which is distorted (stretched or contracted) in some way since we have changed axes again: note that it retains the main features, or dynamics, of the simpler system.

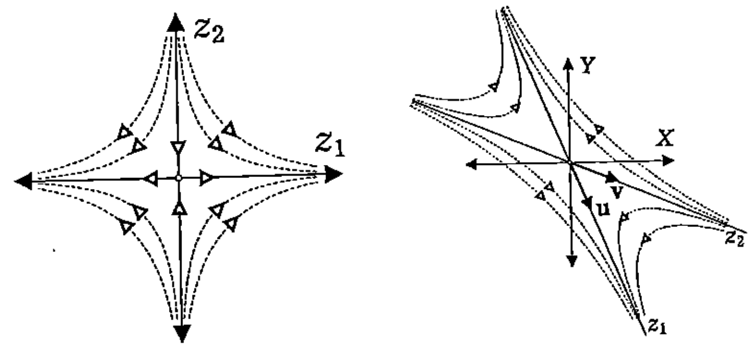

Figure 7.2.2: Sample saddle solution in the eigenvector phase-plane (left figure) and translated to the original phase-plane (right figure).

Equilibrium point classifications

eigenvalues carry the information

For the systems described above, we had the origin $(0, 0)$ as the equilibrium (or critical) point. What we have found, using the techniques of eigenvalues and eigenvectors, is the behaviour of the trajectories in the phase-plane close to this point. The behaviour depends on the eigenvalues (λ_1 and λ_2) since the trajectories can be described by $z_1 = k_1 e^{\lambda_1 t}$ and $z_2 = k_2 e^{\lambda_2 t}$. Clearly different values of λ_1 and λ_2 may result in very different behaviours, and thus each case is dealt with separately below.

the effect of the eigenvalues on the trajectories

The following summarises the relationship between the eigenvalues and the forms of the trajectories:

- Case $\lambda_1 < 0$ and $\lambda_2 < 0$ (eigenvalues real and negative): We have

$$\lim_{t\to\infty} k_1 e^{\lambda_1 t} = 0 \quad \text{and} \quad \lim_{t\to\infty} k_2 e^{\lambda_2 t} = 0$$

  and thus all trajectories approach the equilibrium point at the origin. Such a point is called a *stable node* and is illustrated in Figure 7.2.3.
- Case $\lambda_1 > 0$ and $\lambda_2 > 0$ (eigenvalues real and positive): We have both z_1 and z_2 approaching ∞ (diverging) as t increases and thus all trajectories diverge from the equilibrium point. Such a point is called an *unstable node* (see Figure 7.2.3).

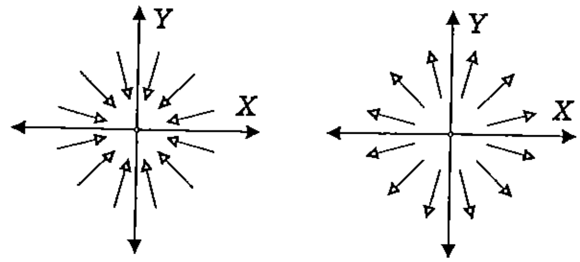

Figure 7.2.3: Trajectory behaviour close to a stable node (left) and an unstable node (right).

- Case $\lambda_1 > 0$ and $\lambda_2 < 0$ (eigenvalues real and of different sign): We have that $z_1 = k_1 e^{\lambda_1 t}$ and $z_2 = k_1 e^{\lambda_2 t}$, so $z_2 \to 0$ and $z_1 \to \infty$ as time increases. The trajectories approach zero along one axis and approach ∞ along the other. Such a point is called a *saddle* or an unstable saddle point and is illustrated in Figure 7.2.4.

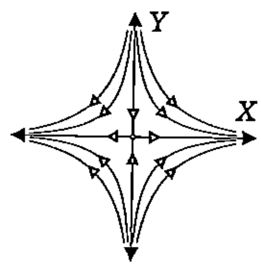

Figure 7.2.4: Trajectory behaviour close to an unstable saddle point.

- Case $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ (complex conjugate eigenvalues with $\alpha \neq 0$ and $\beta \neq 0$): In this case, the solutions can be written in the form $z_1 = e^{\alpha t}\cos\beta t$, $z_2 = e^{\alpha t}\sin\beta t$ and the trajectories spiral around the equilibrium point. If $\alpha < 0$ then they spiral inwards towards the equilibrium point. Such a point is called a *stable focus*. If $\alpha > 0$ then they spiral outwards and away from the equilibrium point. Such a point is called an *unstable focus*. A stable and unstable focus are illustrated in Figure 7.2.5.
- Case λ_1 and λ_2 purely imaginary. In this case, the solutions can be written in the form $z_1 = \cos\beta t$ and $z_2 = \sin\beta t$ and the trajectories form closed loops enclosing the equilibrium point. Such a point is called a *centre* and the solutions are called periodic. A centre is illustrated in Figure 7.2.6.

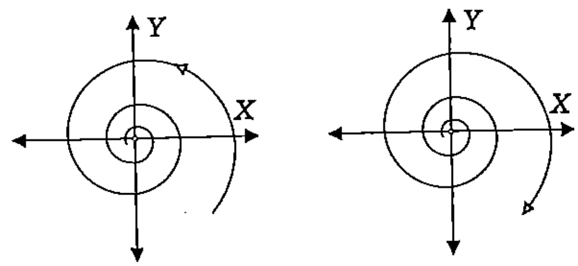

Figure 7.2.5: Trajectory behaviour close to a stable focus (left) and an unstable focus (right).

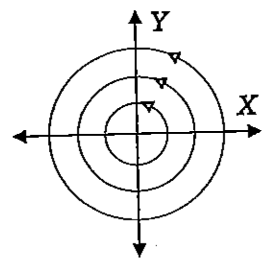

Figure 7.2.6: Trajectory behaviour close to a centre.

seen it all before

We have already come across examples of most of these types of equilibrium points in Chapter 6, where we considered the behaviour of the trajectories in the phase-plane for some basic population models. The power of the above results is that, having located an equilibrium point, we now have the means to predict its type, once we have found the associated eigenvalues.

Summary

- We start with the characteristic equation

$$\lambda^2 - \lambda(a_1 + b_2) + (a_1 b_2 - a_2 b_1) = 0$$

  and solve for λ. This provides us with the eigenvalues.
- Solving the quadratic characteristic equation gives

$$\lambda_1 = \frac{1}{2}p + \frac{1}{2}\sqrt{\Delta}, \qquad \lambda_2 = \frac{1}{2}p - \frac{1}{2}\sqrt{\Delta}$$

  where $p = a_1 + b_2$ is the trace of matrix A, $q = a_1 b_2 - a_2 b_1$ is the determinant of A and $\Delta = p^2 - 4q$ the discriminant of the characteristic equation. The different possible classifications of the equilibrium points are given in Table 7.2.1.

- This can all be displayed in a diagram, Figure 7.2.7, which illustrates the general classifications.

Table 7.2.1: Table showing different classifications of equilibrium points.

Δ	p	q		Equilibrium point
$\Delta > 0$	$p < 0$		$\Rightarrow$	stable node
$\Delta > 0$	$p > 0$		$\Rightarrow$	unstable node
		$q < 0$	$\Rightarrow$	saddle point
$\Delta < 0$	$p < 0$		$\Rightarrow$	stable spiral
$\Delta < 0$	$p = 0$		$\Rightarrow$	centre
$\Delta < 0$	$p > 0$		$\Rightarrow$	unstable spiral

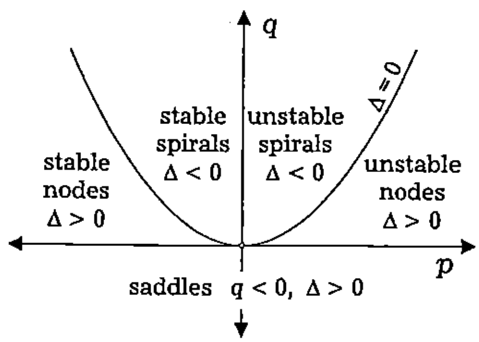

Figure 7.2.7: General classification diagram for equilibrium points using p, q and Δ from the characteristic equation.

Discussion

Note, we have not included the cases where $\Delta = 0$ or $q = 0$. When $\Delta = 0$ the roots are equal and the equilibrium point is a stable/unstable inflected node. Such equilibrium points are called degenerate. When $q = 0$ the system 'equilibrium point' consists of a line and the trajectories are parallel. These special cases are illustrated in Figure 7.2.8. details

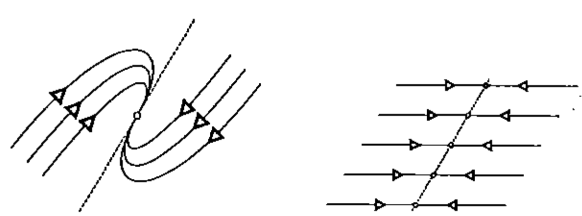

Figure 7.2.8: Trajectory behaviour when $\Delta = 0$ (left) and $q = 0$ (right), illustrated for the stable case when $p < 0$.

attractors and repellers

Stable nodes and spirals are known as attractors, while unstable nodes and spirals are known as repellers. It is important to note that changes in the parameters a_1, a_2, b_1, b_2 can result in very different dynamics. Thus, it is possible to choose the dynamics through controlled variation of the parameters.

dealing with non-zero equilibrium points

We have considered only the case of an equilibrium point at the origin, i.e. $(x_e, y_e) = (0,0)$, but in general this is not so. Suppose we have a system as before, but with $(x_e, y_e) \neq (0,0)$,

$$X' = a_1 X + b_1 Y + c_1,$$
$$Y' = a_2 X + b_2 Y + c_2.$$

Let $\xi = X - x_e$ and $\eta = Y - y_e$ (so that $X = x_e$ implies that $\xi = 0$ and $Y = y_e$ implies that $\eta = 0$). Then

$$\begin{aligned} \xi' = X' &= a_1(\xi + x_e) + b_1(\eta + y_e) + c_1 \\ &= (a_1 x_e + b_1 y_e + c_1) + a_1 \xi + b_1 \eta \\ &= a_1 \xi + b_1 \eta \end{aligned}$$

since $a_1 x_e + b_1 y_e + c_1 = 0$. Similarly

$$\eta' = Y' = a_2(\xi + x_e) + b_2(\eta + y_e) + c_2 = a_2 \xi + b_2 \eta.$$

change of variable

This process is called a change of variable and allows us to transform the original system in X and Y with an equilibrium at (x_e, y_e), to a system with variables ξ and η which has its equilibrium point at the origin. We can now apply the above theory to this system, which is equivalent to the original system of equations.

x is a linear combination of z_1 and z_2

Another assumption we made was that we could express $\mathbf{x}$ as a linear combination of the eigenvectors and the new axes z_1 and z_2. This is a result of the linear algebra presented in Appendix B.1.

U is invertible

Furthermore, we assumed that $U = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$ was invertible, that is, there exists a matrix U^{-1} such that $U^{-1}U = I = UU^{-1}$. This follows as a consequence of Theorem 5 in Appendix B.1.

An alternative approach

a second-order representation

The above approach requires some understanding of linear algebra theory, and this can be avoided if we make the assumption that the solution is an exponential of the form $ke^{\lambda t}$, with λ possibly imaginary. In this case we need to write the coupled linear system

$$X' = a_1 X + b_1 Y,$$
$$Y' = a_2 X + b_2 Y,$$

as a single second-order equation

$$X'' - (a_1 + b_2)X' + (a_1b_2 - a_2b_1)X = 0.$$

(This can be done by differentiating the first equation and then eliminating Y using substitution from the second equation. For further details see Appendix A.5.)

Assuming that there is a solution of the form $X = ke^{\lambda t}$, we can calculate X'' and X' and substitute them into the second-order equation to give

the characteristic equation again

$$k\lambda^2 e^{\lambda t} - (a_1 + b_2)k\lambda e^{\lambda t} + (a_1b_2 - a_2b_1)ke^{\lambda t} = 0,$$

and then

$$\lambda^2 - (a_1 + b_2)\lambda + (a_1b_2 - a_2b_1) = 0.$$

Note that this is the characteristic equation once again. The solutions of this equation are the eigenvalues (λ_1 and λ_2) and from here the general solution is a linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, that is

$$X(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}.$$

We are now in a position to classify the equilibrium points, as before.

Skills developed here:

- Write a system of equations in vector notation, or convert a vector equation into a system of equations.
- Find the trace and determinant of a matrix.
- Calculate the eigenvalues of a matrix.
- From the eigenvalues of a matrix, and for a given equilibrium point, classify the equilibrium point and sketch the trajectory behaviour in the phase-plane close to this point.