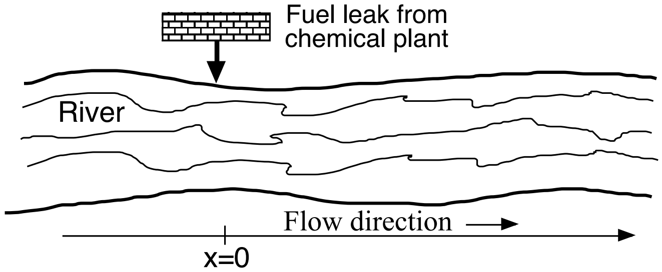
Worksheet

[Refer to handout from last class for help with these exercises.]

Consider the problem of modeling the spread of a pollutant in a river, following an industrial accident that is releasing it into the river at a specific location. The sketch below shows a schematic



Several key modeling concepts pertaining to natural phenomena can be understood with the help of this example. To keep things simple, assume a one-dimensional approximation for the river flow. Let $c(x,t)$ denote the concentration of the pollutant as a function of x and t.

1. **A still river model**: Suppose the flow velocity of the river is zero, and suppose the only mechanism for pollutant transport is diffusion. (E.g., Think of a drop of ink placed gently into a long, narrow channel of still water.)

   (a) Write a partial differential equation for $c(x,t)$ to model this situation.

   (b) Suppose at $t = 0$ we have an initial concentration that looks like an impulse function, e.g., $c(x,0) = e^{-100x^2}$. Sketch a graph of $c(x,0)$, and sketch qualititatively reasonable estimates for $c(x,t)$ at later times. Assume no new pollutant enters the river after $t = 0$.

   (c) Using basic understanding about derivatives, it is possible to guess some solution forms that satisfy the above PDE. For instance, assume $c(x,t) = f(x)\ g(t)$ for some functions f and g. Show that, to satisfy $\partial c/\partial t = \partial^2 c/\partial x^2$, we must have $f(x)\ g'(t) = f''(x)\ g(t)$. Can you guess an exponential g and a trigonometric f that does the job? Can you then extend that to the more general diffusion equation: $\partial c/\partial t = \alpha\ \partial^2 c/\partial x^2$?

   (d) Reflect on your solution in (c) and discuss whether it could explain your qualitative solutions in (b).

2. **A flowing river with no diffusion**: This time we assume a constant flow velocity v for the river, and consider a pollutant that doesn't diffuse. (E.g., Think of a drop of non-diffusing oil placed into a long, narrow tube of flowing water.)

   (a) Write a partial differential equation for $c(x,t)$ that would model this situation.

   (b) Suppose at $t = 0$ we have an initial concentration that looks like an impulse function, e.g., $c(x,0) = e^{-100x^2}$. Sketch a graph of $c(x,0)$, and sketch qualititatively reasonable estimates for $c(x,t)$ at later times. Assume no new pollutant enters the river after $t = 0$.

(c) One way to "guesstimate" a solution for the PDE $\partial c/\partial t = -v\, \partial c/\partial x$, is to notice the solid-body type of motion of the pollutant concentration c. Suppose $c(x, 0) = f(x)$ for some function f. Try to plug into the PDE the guess $c(x, t) = f(x - t)$. Does it work? It should almost! Can you improve the guess, and make it fully work?

(d) Reflect on your solution in (c) and discuss whether it could explain your qualititative solutions in (b).

3. **A more realistic river and pollutant**: Assume the river flows with a constant velocity v, and the pollutant diffuses with constant diffusion coefficient α.

   (a) Write a PDE for $c(x, t)$ to model this situation.

   (b) If the initial pollutant concentration looks like a pulse function, what would you expect it to look like at later times. Show qualititatively reasonable sketches.

4. Run simulations in Sage to explore and verify solutions for each of the above situations.

Moral of the story

An amazingly large number of important and complex natural proccesses – as well as several non-natural processes – are fairly well modeled by a handful of key mathematical forms. In a sense there is something **universal** about these processes, even when they occur in completely different/ unrelated systems, that is clearly visible in their mathematical expressions and properties. We have now seen two examples of such processes:

- **Diffusion**: Kown by various other names such as: conduction (of heat, electricity), brownian motion (of molecules), dissipation, etc.
  But, mathematically it always looks – and acts – the same:
  $$\alpha\, \frac{\partial^2 c}{\partial x^2}, \quad \text{OR} \quad \alpha\, \nabla^2 c \text{ (the more general form).}$$
- **Convection**: Kown by various other names such as: advection, drift, etc.
  Mathematically, it looks like:
  $$v\, \frac{\partial c}{\partial x}, \quad \text{OR} \quad \nabla \cdot (c\, \mathbf{v}) \quad \text{(the more general form).}$$

Homework problems

1. If we consider the one-dimensional diffusion equation at steady-state, it is essentially an ordinary differential equation (not a partial differential equation).

   (a) Show that this is true.

   (b) Consider a domain $0 \leq x \leq 1$, and find the steady-state solution of the one-dimensional diffusion equation for each of the following sets of conditions:
   (i) $c(0) = c(1) = 0$.
   (ii) $c(0) = 0, \quad c(1) = 1$.
   (iii) $c(0) = 0.5, \quad c(1) = 1$.

   I won't tell you what value of diffusion coefficient to use! Why?

   (c) From the above results it is straightforward to deduce the long-term behavior of solutions of the full (unsteady) one-dimensional diffusion equation. Suppose we have the same domain as before ($0 \leq x \leq 1$), and we impose the conditions $c(0) = 0, \; c(1) = 1$, determine the long-term behavior of solutions for each the following initial conditions:
   (i) $c(x, 0) = e^{-100x^2}$.
   (ii) $c(x, 0) = 0$ everywhere, except $c(1, 0) = 0.5$.

   Use a diffusion coefficient value of 1. Justify your answers.

2. If we consider the one-dimensional convection-diffusion equation at steady-state, it is also an ODE (not a PDE).

   (a) Consider a domain $0 \leq x \leq 1$, and find the steady-state solution of the one-dimensional convection-diffusion equation for the following conditions: $c(0) = 0$, $c(1) = 1$. Use $\alpha = v = 1$ for values of diffusion coefficient and convection velocity.

   (b) Repeat the above calculation keeping everything the same, except $v = 10$. Graph and compare the results for $v = 1$ and $v = 10$. What do you observe?
   For a real-world context, it is fairly typical in many applications for the ratio of convection to diffusion to be 10^3 or higher.