How to Prepare

In 3 words: Practice, practice, practice! There's no substitute.

Preface: Many good students prepare poorly for math exams. In my view, the two most critically important ingredients for preparing well are: (1) practice, and (2) constantly testing your understanding throughout the preparation process. In math it is very easy to delude yourself into thinking you understand something, or that you know how to solve a certain type of problem, even when you really don't. Nobody is immune from this – not even math professors! Thus, my advise to students is to get lots of practice and to keep testing your understanding along the way. What follows below are some specific directions you might take while getting your practice.

[1] Work through old quizzes, old exams, and key homework problems.
- best practice is to solve problems from scratch, without first looking at any solutions you may have
- try to add variations & create new problems as you gain comfort & insight

[2] Review your class notes. Pay special attention to:
- solution strategies & examples covered
- theoretical concepts that were indicated to be particularly important
- concepts that you had trouble with

[3] Review the lab exercises.

[4] Rework key homework problems, particularly those you found difficult or didn't get right.

[5] Re-read sections of the textbook that deal with key concepts, as well as those you found difficult or unclear.
I also recommend the "Concept Check" and "True-False quiz" at the end of each chapter.

[6] Work through extra problems, beyond those assigned – especially those that have answers in the textbook. Remember to check the "Review" section at the end of each chapter in the book.

[7] Seek help when you encounter trouble spots and to strengthen any areas that feel weak or murky.

Included topics & syllabus

The test will be based on material from the following sections of the textbook:

* Chapter 2: Sec. 2.5-2.8.
* Chapter 3: Sec. 3.1-3.5.

IMPORTANT NOTE: Although the test won't have any questions directly taken from the earlier materials, you will be expected to know key concepts covered earlier. E.g., the notion of limits and continuity are central to understanding derivatives.

Other key reminders

* Grading will be based on solution process and reasoning – NOT on right answers.

* Remember to bring a graphing calculator, and to use it for verifying your answers and for insights into solution strategies.

* Time-management is important when taking any test. The time-limit will be strictly enforced. Here are some pointers that may be helpful:

  - Budget your time allotment for each problem at the very start.

    E.g., if there are 10 problems, and you have a total of 80 minutes, you can average upto 8 minutes per problem. To play it safe, cut that to 6 or 7 minutes allocated to each problem.

  - Pick the "low-hanging" fruit first

    (i.e., start with the easy problems)

  - If you're stuck, and not making progress for several minutes, move on to the next problem. You can return to the "stuck problem" at the end.

Summary of major shortcuts of differentiation

1. Power rule: $(x^n)' = nx^{n-1}$
2. Product rule: $(fg)' = f'g + fg'$
3. Quotient rule: $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$
4. Chain rule: $\dfrac{df}{dx} = \dfrac{df}{du} \times \dfrac{du}{dx}$
5. Exponential function with base e: $(e^x)' = e^x$
   Important derived rule: $(\ln x)' = \dfrac{1}{x}$
6. Basic trigonometric functions: $(\sin x)' = \cos x, \quad (\cos x)' = -\sin x$
   Important derived rule: $(\tan x)' = \sec^2 x$
7. Implicit differentiation: To differentiate terms written as functions of y, with respect to x, first differentiate with respect to y, then multiply by $\dfrac{dy}{dx}$. E.g., $\dfrac{d}{dx}(xy^2) = y^2 + x\left(2y\dfrac{dy}{dx}\right) = y^2 + 2xy\dfrac{dy}{dx}$
8. Logarithmic differentiation: To differentiate functions that have a variable in the exponent, take "ln" of both sides, apply properties of ln, then implicitly differentiate both sides & solve for the derivative. E.g.,
   $$y = x^{\sqrt{x}} \;\Rightarrow\; \ln(y) = \sqrt{x}\ln(x) \;\Rightarrow\; \frac{y'}{y} = \frac{\sqrt{x}}{x} + \frac{\ln(x)}{2\sqrt{x}}$$
   $$\Rightarrow\; y' = x^{\sqrt{x}}\left[\frac{2 + \ln(x)}{2\sqrt{x}}\right]$$

Other derived rules:

* Trig: $(\sec x)' = \sec x \cdot \tan x$, $(\csc x)' = -\csc x \cdot \cot x$, $(\cot x)' = -\csc^2 x$.

* Inverse trig: $(\sin^{-1} x)' = \dfrac{1}{\sqrt{1-x^2}}, \quad (\cos^{-1} x)' = -\dfrac{1}{\sqrt{1-x^2}},$
  $(\tan^{-1} x)' = \dfrac{1}{1+x^2}.$

Concepts, Definitions and Theorems:

New/significant theoretical concepts are mainly in Chapter 2. I have attempted to identify the primary ones here – be sure you understand them!

1. Limits at infinity. How to define (and find) vertical & horizontal asymptotes of $f(x)$?
2. The tangent line to the graph of a function $f(x)$ at $x = a$. A secant line of the graph of $f(x)$.
3. The average velocity and the instantaneous velocity of an object whose position is recorded by $s(t)$.
4. The derivative $f'(a)$ at $x = a$. How to compute it? What does it mean that $f(x)$ is differentiable at a?
5. The derivative function $f'(x)$. How to compute it? What does it mean that $f(x)$ is differentiable on the interval (a, b)?
6. Key interpretations of the derivative: (i) slope of the tangent to the graph of $f(x)$; (ii) rate of change of f with respect to x.
7. How to eyeball the graph of $f'(x)$ from that of $f(x)$.
   What conditions in the graph of $f(x)$ cause $f'(x)$ to not exist?
8. The second derivative of $f(x)$.
9. What can you tell about $f(x)$ if you know its 1st and 2nd derivative?
10. Theorem: Differentiable implies continuous. If $f(x)$ is differentiable at $x = a$, then it is continuous at $x = a$.
    If $f(x)$ is differentiable everywhere on its domain, then it is also continuous everywhere on its domain.
    Contrapositive Theorem. (not continuous implies not differentiable)
    If $f(x)$ is not continuous at a, then it is not differentiable at a.
    Converse Statement is False! Continuity does not guarantee differentiability. Counterexample?
11. Know how to define the 6 basic trig. functions and their relationship to each other. Understand the unit circle perspective.
12. Understand behavior of $\sin(x)$ & $\cos(x)$ near $x = 0$.
13. Be able to apply all the shortcut rules of differentiation (see previous page for summary).
14. Understand the notion of modeling rates of change via derivatives, as we saw in various labs.

Problem Solving Techniques:

1. How to find the vertical and horizontal asymptotes of any $f(x)$?

   (a) To find vertical asymptotes we must locate the x-values where $f(x)$ approaches $\pm\infty$. In mathematical terms, we must find all points $x = a$ where $\lim_{x\to a} f(x) = +\infty$ or $-\infty$.

   If $f(x)$ is of rational form (ratio of polynomials), these x-values can often be found by determining where its denominator=0. However, be sure to check that the numerator is NOT 0 at the same point. (Can you tell why that is important?)

   If $f(x) = \ln(\text{stuff})$ then there is a vertical asymptote where stuff $= 0$.

   (b) To find the horizontal asymptotes we must evaluate the limits:

   $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

2. How do we find derivatives from the definition? Read carefully if you are being asked to find a specific derivative $f'(a)$ or the whole derivative function $f'(x)$. In each case, you have two choices on how to proceed.

   (a) $f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$. Here a is a constant and x moves towards a. So x will disappear and a will remain in the final result for $f'(a)$.

   (b) $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$. Here a is a constant and h moves towards 0. So h will disappear and a will remain in the final result. This formula is nothing else but formula (a) where x is replaced by $a+h$.

   (c) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$. Here x is viewed as a constant and h moves towards 0. So h will disappear and x will remain in the final result for the derivative function $f'(x)$. This formula is nothing else but formula (b) where a is replaced by x.

3. How do we find equations of tangent lines?

   (a) First find the corresponding derivative $f'(a)$. This will be the slope m of your tangent line.

   (b) Next use the standard straight-line formula $y = mx + b$, with the slope m obtained in part (a). Plug in the point $(a, f(a))$ and solve for *b*. Note that a is a constant, $f(a)$ is also some number, and x, y are variables in the equation.

4. How do we "eyeball" & sketch graphs of $f'(x)$ given the graph of $f(x)$?
(a) Find all x-values where the given function $f(x)$ is not differentiable; at these values $f'(x)$ will not exist. Here are specific reasons why $f'(x)$ may not exist
   * $f(x)$ not defined at $x = a$.
   * $f(x)$ defined but not continuous at $x = a$. Then the contrapositive theorem implies that $f(x)$ is not differentiable at $x = a$.
   * $f(x)$ defined and continuous at $x = a$. However, it is not "smooth" there, i.e., has a cusp or corner.
   * $f(x)$ looks "smooth" at $x = a$. However, it has a vertical tangent there. This will translate into a vertical asymptote of $f'(x)$.

(b) After marking all x-values where $f'(x)$ does not exist (including possible vertical asymptotes, etc.), find all places where the tangent to $f(x)$ is horizontal. Denote these points on the x-axis with solid dots. Your graph of f' must pass through these points.
(c) Next determine the intervals where $f(x)$ is increasing (i.e., positive f'), and where it is decreasing (negative f').
(d) In each such interval, answer the following question: are the tangent slopes increasing or decreasing. Translate this into the appropriate shape of the f' graph.

Problem Solving Notes & Hints on Ch.3:

* There are often 2 (or more) ways to compute the derivative of a given function. Sometimes 1 method is much easier than the others.

E.g., $y = \dfrac{x^3 - 4x^2}{\sqrt[3]{x}}$. Can apply Q.R., or simplify then apply power rule:

$$y = \frac{x^3 - 4x^2}{x^{1/3}} = x^{8/3} - 4x^{5/3} \quad \Rightarrow \quad y' = \frac{8}{3}x^{5/3} - \frac{20}{3}x^{2/3}$$

* Remember to use key trig. identities and the relationship of trig. functions to the sine and cosine. E.g., see pg. 195-196, exercises 12, 29.

* To differentiate functions that contain absolute value items, write as 2 separate equations, clearly specifying domain restrictions on each. Then differentiate each equation as usual. The boundary point(s) where one equa-

tion changes to the other always require special treatment – must look for left/right limits there. Your final answers must clearly indicate domain restrictions as needed.

Some new practice problems

Try the following exercises under simulated testing conditions after you have completed your preparations 75-80% of the way. You'll notice there are way too many problems here to put on a single 80-minute test. So you may take as much time as you like to complete this mock test!

Pg. 164: Concept check: 6, 9, 11, 12, 14, 16.
Pg. 165: True-False: 2, 3, 8, 9, 16.
Pg. 165-167: Exercises: 27, 29, 32, 33, 34, 36, 37(a), 40, 42, 43, 44.

Pg. 248: True-False: Do all of them!
Pg. 248-250: Exercises: 1-7, 11, 28, 31, 38, 42, 51, 53, 54, 55, 61, 64a, 69, 73.

Pointers on what topics I consider important

I try to design tests that assess conceptual knowledge, as well as computational skills. This is easy to say, but hard to successfully implement, or to describe how it should affect your preparation efforts! So, let me just (very briefly) say that I expect you to "deeply" understand what a derivative is, at various levels – math definition, geometric, algebraic, in the context of applications, connections between f' and f. I also hope (and expect) you have thorough understanding of how to compute derivatives using the "short-cut" rules of Chapter 3, together with your algebraic skills.

If I were studying for this test, I would spend most of my time on the following sections: 2.5, 2.7, 2.8, 3.4, 3.5.