Student name:

MATH 180: Calculus A **Test 3**
Spring 2022 **May 10, 2022**

Instructions:

- This is a regular "closed-book" test, and is to be taken without the use of notes, books, or other reference materials. Collaboration or group work is not permitted.
- Cell-phone usage in any form is prohibited for the entire duration of the test. This also applies to any restroom breaks taken during the test.
- Answer all questions on separate paper (not on this sheet!).
- Solve all problems using algebra, except if specifically indicated otherwise. Show all solution steps, give reasons, and simplify your answer to receive full credit.
- The time limit for taking this test is 80 minutes from the scheduled start time.
- This test adds up to 50 points. It contains questions numbered (1) through (7).

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(1) [6 pts.] Consider the Riemann sum $R_n = \sum_{i=1}^{n} \left[\ln\left(1 + \frac{4i}{n}\right) - 2 \right] \frac{4}{n}$.

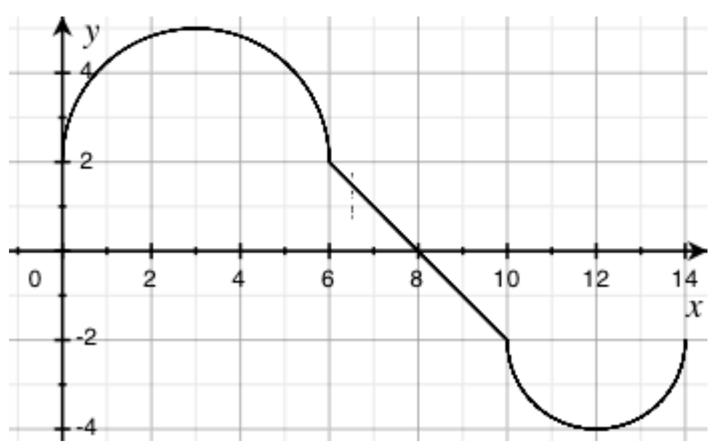
(a) As you know, this approximates a definite integral of the form $\int_a^b f(x)\,dx$. Find $f(x)$, a and b. Show steps and reasons.

(b) Compute the numerical value of R_4.

(2) [6 pts] A piece of wire 20 ft long is to be cut into two pieces. One piece is to be bent into a square, and the other into a circle. Determine the length of each piece so that the sum of the areas is maximized.

(3) [6 pts] Let $y = e^{x|x-2|}$. Find the intervals on which y is increasing, and on which it is decreasing, together with its local extreme values. Credit for correct steps only!

(4) [6 pts.] The graph of $y = f(x)$ shown here consists of line segments and semicircles. Use it to evaluate the following definite integrals by interpreting each in terms of an exact area:



(a) $\int_0^6 f(x)\,dx$

(b) $\int_0^{14} f(x)\,dx$

* In case you're blanking out on it,
area of a semicircle $= \pi r^2/2$, area of a $\Delta = \frac{1}{2}$ base $\times$ height, area of a square $=$ side^2, area of a rectangle $=$ width $\times$ height.

(5) [6 pts.$\times$ 2] Differentiate with respect to x and simplify, showing needed steps and justification:

(a) $y = \dfrac{(2x+1)^3}{x^5\sqrt{x+1}}$ (b) $f(x) = x\sin^{-1}\left(\dfrac{x}{2}\right) + \sqrt{4-x^2}$

Hint: A complete solution to (a) with almost no algebraic mess can be done in 4 lines!

(6) [4 pts.] State the Mean Value Theorem in mathematically precise language, followed by a brief explanation of what it means in everyday language. Include a graph, with labels, to illustrate your claims.

(7) [2.5 pts. $\times$ 4] Give short answers to each of the following as instructed:

(a) Carry out one iteration of Newton's method to find the root of $x^3 - x = 2$ starting from the initial guess $x = 1$.

(b) The area of a square is increasing at a constant rate (with respect to time). Is the side-length increasing at:
(i) a constant rate?
(ii) an increasing rate?
(iii) a decreasing rate?
Give a solid, calculus-based justification for your answer.

(c) Sketch the graph of a function <u>on a closed interval</u> that has no absolute extremes, but does have a critical point. Be sure to include axis labels.

(d) Let $L(x)$ denote the linear approximation of a function $f(x)$ at $x = a$. You, of course, know that $L(x) = f(a) + f'(a)(x-a)$. Let $E(x) = f(x) - L(x)$ denote the error in the linear approximation. What is

$$\lim_{x\to a} \frac{E(x)}{x-a}$$

Hint: Plugin $f(x)$ and $L(x)$ into $E(x)$ and evaluate the limit using any standard method – e.g., algebraic simplification, or other ways ...

End of test

Spring 2022: Calculus A: Test 3 Solutions

[1] (a) The general form of a right Reimann sum for $\int_a^b f(x)\,dx$ is

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x, \quad \text{where } \Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x$$

Comparing this with the given form of R_n, we can see that

$$f(x_i) = \ln\left(1+\frac{4i}{n}\right) - 2 \quad \text{and} \quad \Delta x = \frac{4}{n}$$

From this we can deduce that $x_i = 1 + \frac{4i}{n}$, and $a = 1$, $b = 5$.

Therefore, $f(x) = \ln(x) - 2$, $a = 1$, $b = 5$. The given Reimann sum approximates $\boxed{\int_1^5 [\ln x - 2]\,dx}$

NOTE: Another correct answer that represents this same integral is $\int_0^4 [\ln(x+1) - 2]\,dx$

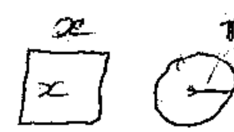
(b) $R_4 = \sum_{i=1}^{4} [\ln(1+i) - 2]\cdot 1 = [\ln(2)-2] + [\ln(3)-2] + [\ln(4)-2] + [\ln(5)-2]$

$$= \ln(2\times 3\times 4\times 5) - 8$$

$$\boxed{= \ln(120) - 8}$$

[2] Let x = side length of square (in feet)

r = radius of circle (feet)



Total area $= x^2 + \pi r^2$

Since the total length is 20': $4x + 2\pi r = 20 \Rightarrow r = \frac{20-4x}{2\pi} = \frac{10-2x}{\pi}$

$$\therefore A(x) = x^2 + \pi\left[\frac{10-2x}{\pi}\right]^2$$

$$\boxed{A(x) = x^2 + \frac{1}{\pi}(100 - 40x + 4x^2)}$$

To maximize A, find its critical points first.

$$A'(x) = 2x + \frac{1}{\pi}(-40 + 8x) = \left(2 + \frac{8}{\pi}\right)x - \frac{40}{\pi} \Rightarrow x = \frac{40/\pi}{(2 + 8/\pi)}$$

Simplifying a bit, we get $x = \frac{40}{2\pi + 8} = \boxed{\frac{20}{\pi+4}}$

The minimum and maximum possible values of x are 0 and $\frac{20}{4} = 5$. To find abs maximum of $A(x)$:

0	$A(0) = \frac{100}{\pi}$
5	$A(5) = 25$
$\frac{20}{\pi+4}$	$A\left(\frac{20}{\pi+4}\right) = 14$

The area is maximized when $x = 0$.

Answer: The maximum area is obtained when the entire wire is used to make the circle.

[3] $y = e^{x|x-2|}$

To find intervals of increase/decrease I'll use a sign graph of y'. Accordingly,

$$y = \begin{cases} e^{x^2-2x}, & \text{if } x \geq 2 \\ e^{-x^2+2x}, & \text{if } x < 2 \end{cases} \implies y' = \begin{cases} (2x-2)e^{x^2-2x}, & \text{if } x > 2 \\ (-2x+2)e^{-x^2+2x}, & \text{if } x < 2 \\ \text{DNE}, & \text{if } x = 2 \end{cases}$$

$y' = 0$ when $x = 1$ and $y' = $ DNE when $x = 2$.

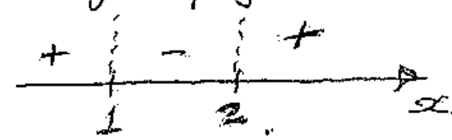
Thus, there are two critical points: $x = 1$ and $x = 2$

When $x > 2$: $y' = 2x - 2 > 0$

When $1 < x < 2$: $y' = -2x + 2 < 0$

When $x < 1$: $y' = \cancel{-2x+2} > 0$

Sign of y'



Answers: y is increasing on $(-\infty, 1)$ and $(2, \infty)$

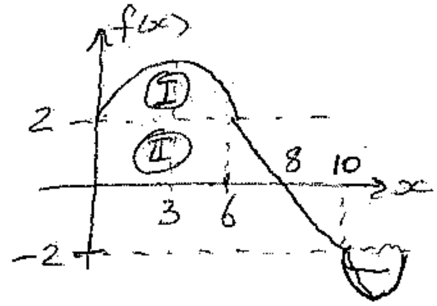
y is decreasing on $(1, 2)$

There is one local maximum @ $(1, e)$

There is one local minimum @ $(2, 1)$

[4] (a) $\int_0^6 f(x)\,dx = $ Area I + Area II

(See sketch)



$$= \frac{\pi(3)^2}{2} + 6 \times 2$$

$$\boxed{\therefore \int_0^6 f(x)\,dx = \frac{9\pi}{2} + 12}$$

(b) $\int_0^{14} f(x)\,dx = \int_0^6 + \int_6^{10} + \int_{10}^{14}$

We already have $\int_0^6 f(x)\,dx$ from part (a).

And, from the symmetry shown in the sketch $\int_6^{10} f(x)\,dx = 0$

$$\int_{10}^{14} f(x)\,dx = -(\text{area of } \square + \text{area of } \cup)$$
$$= -\left(4 \times 2 + \pi \frac{2^2}{2}\right) = -(8 + 2\pi)$$

$$\therefore \int_0^{14} f(x)\,dx = \left[\frac{9\pi}{2} + 12\right] + 0 - (8 + 2\pi)$$

Answer: $\boxed{\int_0^{14} f(x)\,dx = \frac{5\pi}{2} + 4}$

[5] (a) $y = \dfrac{(2x+1)^3}{x^5\sqrt{x+1}}$. To find $\dfrac{dy}{dx}$, apply ln to both sides.

$$\ln y = \ln\left[\frac{(2x+1)^3}{x^5(x+1)^{1/2}}\right] = 3\ln(2x+1) - \left[5\ln x + \tfrac{1}{2}\ln(x+1)\right]$$

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{(2x+1)}(2x+1)' - \left[5\left(\frac{1}{x}\right) + \frac{1}{2}\left(\frac{1}{x+1}\right)\cdot(x+1)'\right]$$

$$= \frac{6}{2x+1} - \frac{5}{x} - \frac{1}{2(x+1)}$$

$$\therefore \boxed{\frac{dy}{dx} = \left(\frac{(2x+1)^3}{x^5\sqrt{x+1}}\right)\left[\frac{6}{2x+1} - \frac{5}{x} - \frac{1}{2(x+1)}\right]}$$

* simplifying this further doesn't seem to give anything "simpler," so we can leave it here!

(b) $f(x) = x\cdot\sin^{-1}\left(\frac{x}{2}\right) + \sqrt{4-x^2}$

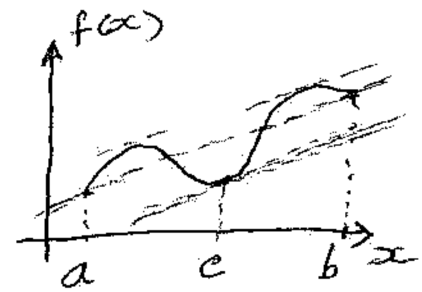
$$\frac{df}{dx} = x\cdot\frac{d}{dx}\sin^{-1}\left(\frac{x}{2}\right) + \sin^{-1}\left(\frac{x}{2}\right)\cdot\frac{d}{dx}x + \frac{d}{dx}(4-x^2)^{1/2}$$

$$= x\cdot\frac{1}{\sqrt{1-(x/2)^2}}\cdot\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{x}{2}\right)\cdot 1 + \frac{1}{2\sqrt{4-x^2}}\cdot(-2x)$$

$$= \frac{x}{2\sqrt{4-x^2}}\times 4 + \sin^{-1}\left(\frac{x}{2}\right) - \frac{x}{\sqrt{4-x^2}}$$

$$\therefore \boxed{\frac{df}{dx} = \sin^{-1}\left(\frac{x}{2}\right)}$$

[6] The MVT says: Suppose f is a differentiable function on the interval $[a, b]$. Then there exists some point c between a and b such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.



In everyday language: If the function is differentiable on the interval (i.e, its graph is smooth), then there is at least one point between a and b where the tangent line has the same slope as the secant line that joins $(a, f(a))$ and $(b, f(b))$.

[7] (a) Given $x^3 - x = 2$. Let $f(x) = x^3 - x - 2$.

Then $f'(x) = 3x^2 - 1$, and $f'(1) = 3 - 1 = 2$

The linear approx at $x=1$ is: $L(x) = f(1) + f'(1)(x-1) = -2 + 2(x-1)$

To find the root, set $L(x) = 0$: $0 = -2 + 2(x-1) \Rightarrow \boxed{x = 2}$

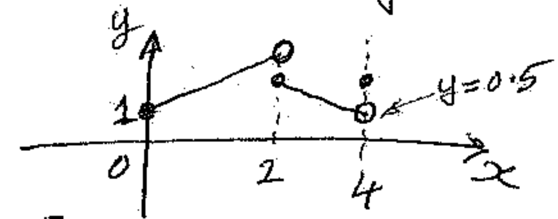
At the end of one Newton iteration, $x_1 = 2$

(b) $\boxed{x}\,x$. $A(x)=x^2 \Rightarrow \frac{dA}{dt}=2x\cdot\frac{dx}{dt} \Rightarrow \frac{dx}{dt}=\left(\frac{1}{2x}\right)\frac{dA}{dt}$

Since area is increasing at a constant rate, we know $\frac{dA}{dt}$ = positive constant. Thus, as x increases, $\frac{dx}{dt}$ decreases.

Answer: The side length increases at a decreasing rate.

(c) Graph of function on closed interval that has a critical point but no abs. extremes:



y is defined on the closed interval $[0,4]$

There is a critical point at $x=2$, since y' = DNE at $x=2$.

y has no absolute minimum or maximum.

(d) Let $L(x)$ denote linear approx of $f(x)$ at $x=a$.

Then $L(x)=f(a)+f'(a)(x-a)$

$E(x)=f(x)-L(x)=f(x)-f(a)-f'(a)(x-a)$

And $\frac{E(x)}{(x-a)}=\frac{f(x)-f(a)-f'(a)(x-a)}{x-a}=\frac{f(x)-f(a)}{x-a}-f'(a)$

$\therefore \lim_{x\to a}\frac{E(x)}{x-a}=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}-f'(a)\right]=f'(a)-f'(a)$

Answer: $\boxed{\lim_{x\to a}\frac{E(x)}{x-a}=0}$

Grading Notes

[1] (a) = 4 points, (b) = 2 points

(a) 1 pt each for correct: a, b, $f(x)$, steps, 1 pt = correctly sum values

(b) 0.5 pt = get $\Delta x=1$, 0.5 pt = get correct x-values, 1 pt = correctly sum values

[2] 1.5 pt = introduce unknowns & setup eqn. for total length/perimeter

1.5 pt = find correct objective function with 1 unknown

1 pt = correct derivative of A; 1.5 pt = find C.P. and make table that shows correct values of A at C.P and end points; 0.5 pt = correct answer

[3] 0.5+0.5 pt = correct derivative of neg + positive part of abs value fn.

1 pt = state correct domain where each part is valid

1 pt = explain/show/justify each of 2 critical points

1 pt = correct sign graph of y'

1 pt = interpret/show why & where y is increasing/decreasing

1 pt = show where & why local extreme occurs.

[4] (a) = 3 points, (b) = 3 points

(a) 1pt = attempt to find sum of area of △ and ▭.
2 pt = do it correctly and add areas to show correct answer

(b) 1 pt = correctly split $\int_0^4$ into reasonable sub-problems.
1pt = evaluate the sub-problems correctly & get numerical values
1pt = add numerical values correctly to state final answer.

[5] (a) 1 pt = attempt to apply ln on both sides
2 pt = do it correctly and simplify to: $\ln y = 3\ln(2x+1) - [5\ln x + \frac{1}{2} \cdots]$
2pt = take derivatives on both sides correctly and simplify
1pt = multiply through by y, and reverse sub. the expression for y

[6] (b) 3pt = correct derivative of $x \cdot \sin^{-1}\left(\frac{x}{2}\right)$ and simplify
2pt = correct derivative of $\sqrt{4-x^2}$ and simplify.
1 pt = add correctly and get answer

[6] 1 pt = correct graph to illustrate statement of MVT
1+1 pt = correct hypotheses + math form of conclusion
1pt = explain in ordinary language, using secant/tangent etc.

[7] (a) 1 pt = find $f(1)$ and $f'(1)$ correctly
1pt = set up expression for $L(x)$ or formula for Newtn. iter.
0.5pt = solve and get $x = 2$.

(b) 1pt = define $A(x) = x^2$, differentiate and get $\frac{dA}{dt} = 2x \cdot \frac{dx}{dt}$
0.5pt = express dx/dt as $(1/2x)\frac{dA}{dt}$ and argue that dx/dt decreases
1pt = correct answer (incl. by intuition)

(c) 1 pt = graph has no abs extremes; 0.5pt = it has C.P.,
0.5pt = closed interval; 0.5pt = axis labels.

(d) 1 pt = correctly plugin $f(x)$ and $L(x)$ into $\frac{E(x)}{x-a}$.
1.5pt = figure out how to get correct limit as $x \to a$.