Student name:

MATH 180: Calculus A | **Test 2**
Spring 2022 | **April 5, 2022**

Instructions:

- This is a regular "closed-book" test, and is to be taken without the use of notes, books, or other reference materials. Collaboration or group work is not permitted.
- Cell-phone usage in any form is prohibited for the entire duration of the test. This also applies to any restroom breaks taken during the test.
- Answer all questions on separate paper (not on this sheet!).
- Solve all problems using algebra, except if specifically indicated otherwise.
  Show all solution steps, give reasons, and simplify your answer to receive full credit.
- The time limit for taking this test is 80 minutes from the scheduled start time.
- This test adds up to 50 points. It contains questions numbered (1) through (7).

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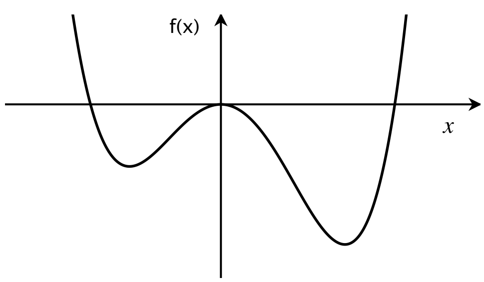
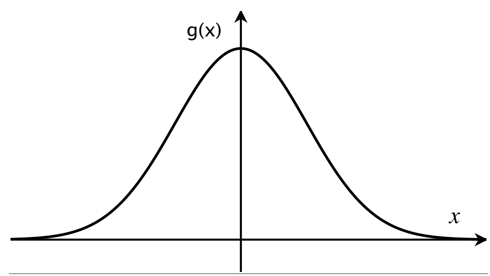
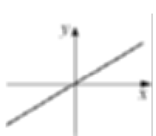
(1) [4 pts.] The graphs of two different functions $f(x)$ and $g(x)$ are shown below:



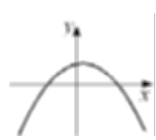


For each function, identify the graph of its derivative from the following (give reasons):

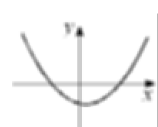
(A)



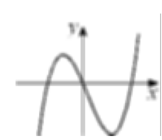
(B)



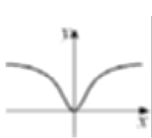
(C)



(D)

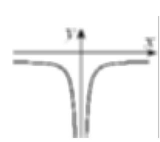


(E)

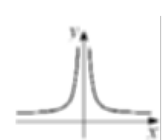


(F)

(G)



(H)



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(2) [4 pts.] Indicate true or false for each, and give reason(s) to justify your answer:

(a) If f is differentiable at $x = a$, then $\lim_{x\to a} f(x)$ exists.

(b) If the $\lim_{x\to a} f(x)$ exists, then f is differentiable at $x = a$.

(3) [4 pts.] Suppose $h(x) = f(g(x))$. Find $h'(2)$, given the following: $g(2) = -1$, $g'(2) = -2$, $f(-1) = 0$, $f'(-1) = 4$. Show steps.

(4) [6 pts.×3] For each of the following, find $\frac{dy}{dx}$ and simplify:

(a) $y = \dfrac{x^2}{2x-1}$ (b) $y = \sqrt{x}\, e^{\sin x}$

(c) $x = \cos(t^2 - \cos t)$, $y = \sin(3t)$

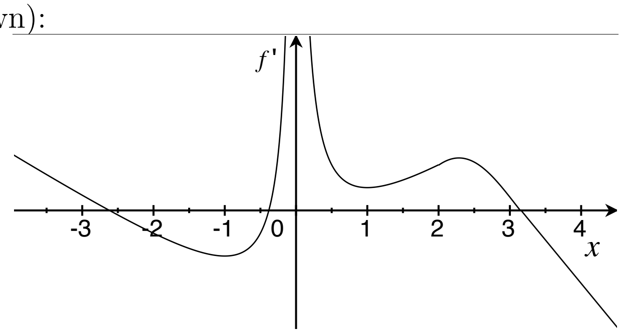
(5) [6 pts.] Let $f(x) = \dfrac{x^2 - 4x + 3}{x^3 + x^2 - x - 1}$. Explain what limits you need to evaluate to find all the vertical and horizontal asymptotes of f. Evaluate those limits and find the asymptotes.

(6) [6 pts.] Give brief answers to each of the following (unrelated) questions, as instructed.

(i) A descending aircraft approaches the airport where it is to land, in a straight flight path. Suppose $h(x)$ denotes its altitude in feet, as a function of distance from the airport in miles. Give a quantitatively precise explanation of what $h'(20) = -300$ means in this context, and give its units.

(ii) Suppose $h(x) = f(x) \cdot g(x)$. Use each of the two limit-based definitions of the derivative to set up formulas for finding h' at $x = a$. You don't need to actually evaluate the limit and/or simplify any further.

(7) [8 pts.] Shown here is the graph of f', the derivative of some continuous function f. Based on this graph, answer the following questions (assume the graph of f' continues to infinity on both ends in the direction shown):



(a) On what interval(s) is f increasing, and on what interval(s) is it decreasing? Give reasons.

(b) At what x-values does f have local minimum and maximum values? Reason?

(c) What kind of concavity does f have when $-1 < x < 0$? Reason?

(d) Is there enough information here to tell whether $f(0)$ is less than, or greater than, $f(2)$? Explain your answer.

End of test

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[1] $f'(x)$ = Graph D. Reason: f has horizontal tangents at 3 points, where f' must equal 0. Only graph D has 3 places where $f'=0$. In addition, regions of increasing f coincide with positive regions in graph D.

$g'(x)$ = Graph F

$\longrightarrow$ Reason: Since g is increasing on $(-\infty, 0)$ and decreasing on $(0,\infty)$, g' must be positive on $(-\infty,0)$ and negative on $(0,\infty)$. Only F has this behavior.

[2]

(a) True. Reason: Since f is differentiable at $x=a$, it must be continuous there, by a theorem. Continuity implies $\lim_{x\to a} f(x)$ exists.

(b) False. Reason: Here is a counterexample: Let $f(x)=|x|$. At $x=0$, f is continuous. Thus, $\lim_{x\to 0} f(x)$ exists. But f is not differentiable at $x=0$.

[3] Given $h(x) = f(g(x))$.

By the chain rule: $h'(x) = f'(g(x))\cdot g'(x)$

$$\therefore h'(2) = f'(g(2))\cdot g'(2) = \underbrace{f'(-1)\cdot(-2) = (4)\cdot(-2)}_{\text{These values are from the given data in the problem.}}$$

$$\boxed{\therefore h'(2) = -8}$$

[4] (a) $y = \dfrac{x^2}{2x-1}$

Using quotient rule: $y' = \dfrac{(2x-1)(x^2)' - x^2(2x-1)'}{(2x-1)^2}$

$$\Rightarrow y' = \frac{(2x-1)(2x) - x^2(2)}{(2x-1)^2} = \frac{4x^2-2x-2x^2}{(2x-1)^2}$$

$$\therefore \boxed{y' = \frac{2x^2-2x}{(2x-1)^2} = \frac{2x(x-1)}{(2x-1)^2}}$$

[4] (b) $y = \sqrt{x}\cdot e^{\sin(x)}$

Use product rule + chain rule: $y' = \sqrt{x}\left[e^{\sin(x)}\right]' + e^{\sin(x)}(\sqrt{x})'$

$$\Rightarrow y' = \sqrt{x}\left[e^{\sin(x)}\cdot(\sin x)'\right] + e^{\sin(x)}\cdot\left(\frac{1}{2\sqrt{x}}\right)$$

$$\therefore \boxed{y' = e^{\sin(x)}\left[\sqrt{x}\cdot\cos(x) + \frac{1}{2\sqrt{x}}\right] = \frac{e^{\sin(x)}}{2\sqrt{x}}\left[2x\cdot\cos x + 1\right]}$$

[4] (c) $x = \cos(t^2 - \cos t)$, $y = \sin(3t)$

The derivative of parametric curves is: $\dfrac{dy}{dx} \equiv \dfrac{dy/dt}{dx/dt}$

$$y = \sin(3t) \Rightarrow \frac{dy}{dx} = \cos(3t)\cdot\frac{d}{dt}(3t) = 3\cdot\cos(3t)$$

$$x = \cos(t^2 - \cos t) \Rightarrow \frac{dx}{dt} = -\sin(t^2 - \cos t)\cdot\frac{d}{dt}(t^2 - \cos t)$$
$$= -\sin(t^2 - \cos t)\cdot(2t + \sin t)$$

$$\boxed{\therefore \frac{dy}{dx} = -\frac{3\cdot\cos(3t)}{\sin(t^2 - \cos t)\cdot(2t + \sin t)}}$$

[5] $$f(x) = \frac{x^2 - 4x + 3}{x^3 + x^2 - x - 1}$$

* Horizontal asymptotes: If $\lim_{x\to\pm\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of f.
* Vertical asymptotes: If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $x = a$ is a vertical asymptote. [The limit could be 1-sided]

Factor denominator to get candidate points where $f \to +\infty$ or $-\infty$.

$$x^3 + x^2 - x - 1 = x(x^2-1) + (x^2-1) = (x^2-1)(x+1) = (x-1)(x+1)^2$$

Thus, $x = \pm 1$ may be vertical asymptotes.

check numerator: When $x = 1$, num $= 0$; when $x = -1$, num $= 8$.

$\Longrightarrow$ Only one vertical asymptote: $\boxed{\text{at } x = -1}$

* Horizontal: $$\lim_{x\to\infty} \frac{x^2 - 4x + 3}{x^3 + x^2 - x - 1} = \lim_{x\to\infty} \frac{1/x - 4/x^2 + 3/x^3}{1 + 1/x - 1/x^2 - 1/x^3} = \frac{0}{1}$$

Thus, horizontal asymptote is: $\boxed{y = 0}$ (i.e, the x-axis)

NOTE: The limit as $x \to -\infty$ is the same for rationals.

[6] (i) $h'(20) = -300$ means: When the aircraft is 20 miles from the airport, its altitude is decreasing at the rate of 300 feet per mile.

(ii) $h(x) = f(x)\cdot g(x)$.

$$h'(a) = \lim_{x\to a} \frac{f(x)\cdot g(x) - f(a)\cdot g(a)}{x - a} = \lim_{h\to 0} \frac{f(a+h)\cdot g(a+h) - f(a)g(a)}{h}$$

[7] (a) f is ↑ on the intervals: $(-\infty, -2.5)$, $(-0.4, 0)$ and $(0, 3.2)$.
Reason: $f' > 0$ those intervals
f is ↓ on: $(-2.5, -0.4)$ and $(3.2, \infty)$. Reason: $f' < 0$ there.

(b) f has local minimum at: $x = -0.4$
Reason: f' crosses the x-axis from negative to positive there.
f has local maximum at: $x = -2.5$ and $x = 3.2$
Reason: f' crosses x-axis from positive to negative there.

(c) f is concave up on $-1 < x < 0$, because f' is increasing on that interval.

(d) Yes. $f(0)$ must be less than $f(2)$.
Reason: Between $x = 0$ and $x = 2$, f' is always positive, which means f is continuously increasing. Therefore, $f(2)$ will have to be larger than $f(0)$.

Grading notes:

[1] 2 points each for correct f' and g'
For each case, 1.5 pt for correctly identifying, and 0.5 pt for reason.

[2] (a) = 2 points, (b) = 2 points
For each, 50/50 split between answer & reason.

[3] 2 pt = get $h'(x) = f'(g(x)) \cdot g'(x)$; 2 pt = plug in right values & get answer.

[4] (a) 2 pt = Know/show QR, with correct quantities plugged into right places;
3 pt = correct derivatives in numerator of QR.
1 pt = clean up and get final result

(b) 1 pt = plug into product rule correctly
1.5 pt = correctly differentiate $\sqrt{x}$; 2.5 pt = differentiate $e^{\sin(x)}$
1 pt = simplify to some reasonable form

(c) 1 pt = Know/show that $dy/dx = (dy/dt)/(dx/dt)$
2+2 pt = correctly differentiate $x(t)$ + $y(t)$.
1 pt = correctly put together and get dy/dx

[5] 1+1 pt = explain limit-based defn. of H.A. + V.A.; 2 pt = correct work with vertical
2 pt = correct process & result for horizontal (−0.5 if $x = 1$ is included in V.A.)

[6] 3 pts each for (i) and (ii)
For (i): 0.5 pt. for units; other than that, must mention (a) rate of change, (b) with respect to distance, (c) when distance is 20 mi.
For (ii): 1.5 pt for each correct limit-based form.

[7] 2 pts each for (a)−(d). Generally, 50/50 split for answer/reason.