Student name:

MATH 180: Calculus A **Test 1**
Spring 2022 **March 1, 2022**

Instructions:

- This is a regular "closed-book" test, and is to be taken without the use of notes, books, or other reference materials. Collaboration or group work is not permitted.
- Cell-phone usage in any form is prohibited for the entire duration of the test. This also applies to any restroom breaks taken during the test.
- Answer all questions on separate paper (not on this sheet!).
- Solve all problems using algebra, except if specifically indicated otherwise. Show all solution steps, give reasons, and simplify your answer to receive full credit.
- The time limit for taking this test is 80 minutes from the scheduled start time. Please turn in your test promptly when time is called to avoid late penalties.
- This test adds up to 50 points. It contains questions numbered 1 through 6.

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(1) [6 pts.] Sketch the graph of a function f with the following properties

$$\lim_{x\to -4} f(x) = -\infty,\ \lim_{x\to 0^-} f(x) = -1,\ f(0) = 0,\ \lim_{x\to 0^+} f(x) = 1,\ \lim_{x\to\infty} f(x) = -3$$

Graph must include detailed labels, and indicate open/closed intervals as needed.

(2) [6 pts.] Find the inverse of the function: $y = \ln(x-3) - \ln(x+3)$

(3) [6 pts.] (a) State and explain, with the help of a sketch, what the Intermediate Value Theorem says. Explanation doesn't have to be long, but it must be clear and mathematically precise.

(b) Show that the equation $(x-1)e^x = 3 - x$ has a solution by applying the IVT.

(4) [6 pts. each $\times$ 3] Evaluate the following limits using algebra and showing all steps. If a limit fails to exist, be sure to determine whether it is ∞, $-\infty$, or some other form of DNE.

(a) $\displaystyle\lim_{x\to 3}\left(\frac{3}{x-3} - \frac{2x^2}{x^2-9}\right)$

(b) $\displaystyle\lim_{x\to 4}\frac{16-x^2}{\sqrt{x}-2}$

(c) $\displaystyle\lim_{x\to 1}\frac{4x^2-4}{|x-1|}$

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(5) [7 pts.] Given the function

$$f(x) = \begin{cases} x^2 + 4x + 5, & \text{if } x < -2 \\ \frac{1}{2}x, & \text{if } |x| < 2 \\ \sqrt{x-2}, & \text{if } x > 2 \end{cases}$$

Find all the values of x where f is continuous. Justify all claims using the mathematical definition of continuity or by using relevant theorems.

(6) [7 pts.] The growth rate of the world's human population is thought to be approximately logistic. We will study logistic models later in the semester. A problem with logistic models is that they require an estimate of the earth's carrying capacity, which is unknown. The following function is a model of the population based on a carrying capacity of 50 billion people, and based on actual data from the year 1990, when the population was 5.3 billion (t denotes the number of years since 1990, and P the population in billions)

$$P(t) = \frac{50}{1 + 8.4\, e^{-0.014t}}$$

(a) Using this model, find the average change in world population over the time period beginning with the year 2022 and lasting: 5 years, 1 year, and 0.5 year. In other words, we want to compute three separate averages here. Show key steps.
(b) From these averages, estimate the slope of the tangent line to the graph of $P(t)$ when the year is 2022. Include units of the slope, and explain why your estimate is reasonable.

End of test

Spring 2022: Calculus A: Test 1 solutions

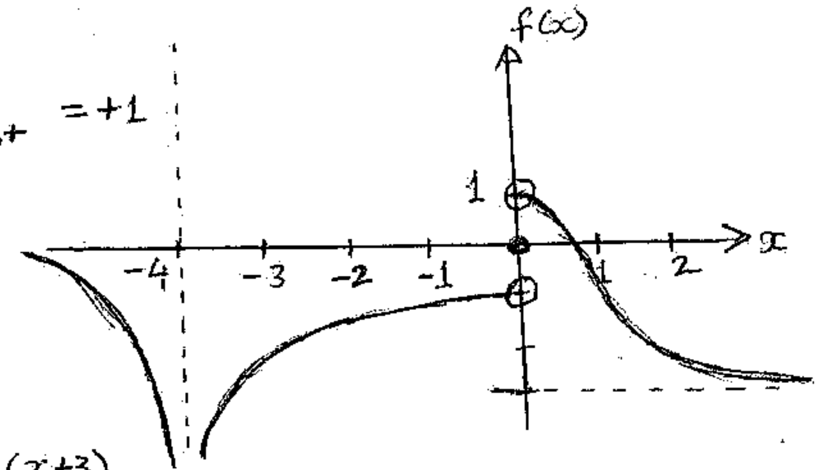
[1]

$\lim_{x\to -4} f(x) = -\infty$

$\lim_{x\to 0^-} f(x) = -1$, $\lim_{x\to 0^+} = +1$

$f(0) = 0$

$\lim_{x\to\infty} = -3$



[2] Want to find the inverse of $y = \ln(x-3) - \ln(x+3)$

Switch roles of x, y: $x = \ln(y-3) - \ln(y+3)$

Solve for y in terms of x:

$$x = \ln\left[\frac{y-3}{y+3}\right] \Rightarrow e^x = \frac{y-3}{y+3}$$

Multiply by $(y+3)$: $e^x(y+3) = y-3$

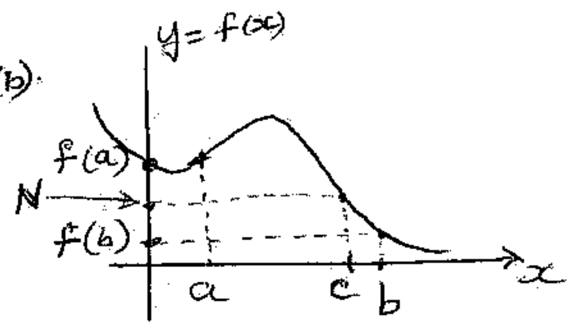
Expand & move all y terms to one side: $3 + 3e^x = y - ye^x$

Factor & divide: $y = \dfrac{3+3e^x}{1-e^x}$

Answer: The inverse is $\boxed{y = \dfrac{3(1+e^x)}{1-e^x}}$

[3] (a) The intermediate value theorem says:

Suppose f is a continuous function on the interval $[a, b]$, and suppose $f(a) \neq f(b)$. Then $f(x)$ takes on every y-value between $f(a)$ and $f(b)$ for some x between a and b.



In more formal terms: If N is any number between $f(a)$ and $f(b)$, then there is some number c between a and b where $f(c) = N$.

(b) Let $f(x) = (x-1)e^x + x - 3$

Since $(x-1)$, e^x, x, and 3 are continuous for all x, so is $f(x)$ continuous (by continuity theorems).

We have: $f(0) = -4$ and $f(2) = e^2 - 1 > 0$

Since f is continuous on $[0, 2]$, by the IVT $f(c) = 0$ for some c between 0 and 2. It follows that the given equation has a root at $x = c$.

[4] (a) $\lim_{x \to 3} \left(\frac{3}{x-3} - \frac{2x^2}{x^2-9} \right)$. Plug in gives 0 denominator, so try other algebraic methods!

$$\frac{3}{x-3} - \frac{2x^2}{x^2-9} = \frac{3(x+3) - 2x^2}{(x+3)(x-3)} = \frac{-2x^2+3x+9}{(x+3)(x-3)} = \frac{\cancel{(x-3)}(-2x-3)}{(x+3)\cancel{(x-3)}}$$

$\therefore \lim_{x \to 3}(\text{original}) = \lim_{x \to 3} \frac{-2x-3}{x+3} = -\frac{9}{6} = \boxed{-\frac{3}{2}}$ Answer

4 (b) $\lim_{x \to 4} \frac{16-x^2}{\sqrt{x}-2}$ Plugin gives $\frac{0}{0}$ ☹ Try to rationalize

$$\frac{16-x^2}{\sqrt{x}-2} = \frac{(4-x)(4+x)}{\sqrt{x}-2} = \frac{\cancel{(2-\sqrt{x})}(2+\sqrt{x})(4+x)}{\cancel{(\sqrt{x}-2)}} = -(2+\sqrt{x})(4+x)$$

$\therefore \lim_{x \to 4}(\text{original}) = \lim_{x \to 4} -(2+\sqrt{x})(4+x) = \boxed{-32}$ Answer

4 (c) $\lim_{x \to 1} \frac{4x^2-4}{|x-1|}$ consider left/right limits, since the abs value changes sign as $x \to 1$.

$$\frac{4x^2-4}{|x-1|} = \begin{cases} \frac{4x^2-4}{x-1}, & \text{if } x > 1 \\ \frac{4x^2-4}{1-x}, & \text{if } x < 1 \end{cases}$$

$$\lim_{x \to 1^-} \frac{4x^2-4}{|x-1|} = \lim_{x \to 1^-} \frac{4(x^2-1)}{1-x} = \lim_{x \to 1^-} \frac{4\cancel{(x-1)}(x+1)}{-\cancel{(x-1)}} = -8$$

$$\lim_{x \to 1^+} \frac{4x^2-4}{|x-1|} = \lim_{x \to 1^+} \frac{4(x+1)(x-1)}{x-1} = \lim_{x \to 1^+} 4(x+1) = 8$$

Since the left/right limits are unequal, the limit $\boxed{\text{DNE}}$

[5] NOTE: There was a typo in this problem: the middle piece should have been defined on $|x| \le 2$, instead of $|x| < 2$. The following solution is for the problem as written (with typo).

Consider the 3 parts of the domain of f: ① $x < -2$, ② $-2 < x < 2$, and ③ $x > 2$. On each part of the domain, f is defined by a polynomial or square root function, which are continuous by theorems on continuity. Thus f is continuous for all x for which it is defined.

Extra info: Had f been defined at $x = \pm 2$, we would have checked whether $\lim_{x \to a} f(x) = f(a)$ for continuity at $x = a$.

[6] The given population model is: $P(t) = \dfrac{50}{1 + 8.4\, e^{-0.014t}}$

t = # of years since 1990
P = population in billions

(a) The year 2022 is 32 years after 1990. Thus $t = 32$ when year = 2022.

Avg. change between 2022–2027: $\dfrac{P(37) - P(32)}{5} = \dfrac{8.3278 - 7.8532}{5} = \boxed{0.0949 \text{ billion/year}}$

Bet. 2022–2023: $\dfrac{P(33) - P(32)}{1} = 7.9464 - 7.8532 = \boxed{0.0931 \text{ billion/year}}$

Bet. 2022–2022.5: $\dfrac{P(32.5) - P(32)}{0.5} = \dfrac{7.8997 - 7.8532}{0.5} = \boxed{0.0929 \text{ billion/year}}$

(b) I estimate the slope of the tangent line is about 0.0925 billion per year

Looking at the trends in the average slopes, it appears that the tangent line will have a slope somewhat smaller than the ~~best~~ closest average we found above, which was 0.0929. Thus, 0.0925 should be a reasonable estimate.

Grading Notes:

[1] 1 pt. for each of 5 requirements: $\lim_{x\to -4}$, $\lim_{x\to 0^-}$, $\lim_{x\to 0^+}$, $f(0)$, $\lim_{x\to\infty}$
1 pt. for axis labels + clear open/closed circles.

[2] 1+1 pt = apply log property + write as $e^y = (x-3)/(x+3)$
3 pt = do algebra and get $x = (3+3e^y)/(1-e^y)$; 1 pt = reverse x, y, and express correct answer

[3] (a) = (b) = 3 points each.
(a) 1 pt = correct hypotheses; 1.5 pt = correct conclusion, 1 pt = sketch
(b) 1 pt = turn equation into function; 0.5 pt = check continuity;
1.5 pt = obtain two x-values where f switches sign & draw conclusion

[4] (a) 1 pt = attempt common denom; 2 pt = do it correctly;
2 pt = factor & cancel; 1 pt = plug in and get correct answer
(b) 1 pt = attempt to rationalize; 2 pt = do it correctly;
2 pt = cancel needed factors; 1 pt = plug in and get answer
(c) 1 pt = attempt to find left/right limits; 2+2 pt = do each correctly
1 pt = conclude correct answer.

[5] 2 pt = state or show awareness of math defn of continuity
5 pt = either my solution above, or correct discussion & conclusion

[6] (a) = 5.5 pt, (b) = 1.5 pt.
(a) 1 pt = calculation details for at least one; 3.5 pt = 1 pt for each correct answer to average; 1 pt = include correct units.