Quiz 4 - 3/08/2022

(I) Use the definition of derivative to find an expression for the derivative of $f(x) = 1/\sqrt{x}$ at $x = a$.

(NOTE: The goal of this question is to assess your understanding of the definition of the derivative, and your ability to use it. Thus, if you know other methods of finding the correct answer, it would be pointless using them here!)

(II) The total cost (in dollars) of repaying a student loan at an interest rate of $r\%$ per year is given by the function $f(r)$. Interpret, with correct units, the meaning of $f'(5) = 2000$.

Solution

(I) By definition, the derivative of $f(x)$ at any $x = a$ is: $f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$.

Plugin the given $f(x)$ and get $f'(a) = \lim_{x\to a} \dfrac{(1/\sqrt{x}) - (1/\sqrt{a})}{x - a}$.

Must try to cancel $(x - a)$ to evaluate this limit.

Rationalize: $$\frac{(1/\sqrt{x}) - (1/\sqrt{a})}{x - a} = \frac{(1/\sqrt{x} - 1/\sqrt{a})}{(x - a)} \cdot \frac{(1/\sqrt{x} + 1/\sqrt{a})}{(1/\sqrt{x} + 1/\sqrt{a})} = \frac{1/x \ - \ 1/a}{(x - a) \cdot (1/\sqrt{x} + 1/\sqrt{a})}$$

$$= \frac{a - x}{x \cdot a \cdot (x - a) \cdot (1/\sqrt{x} + 1/\sqrt{a})} = \frac{-1}{x \cdot a \cdot (1/\sqrt{x} + 1/\sqrt{a})}$$

Plug in $x = a$: $f'(a) = \left[\dfrac{-1}{a \cdot a \cdot (1/\sqrt{a} + 1/\sqrt{a})}\right] = \dfrac{-1}{2a\sqrt{a}}$.

Answer: $\boxed{f'(a) = \dfrac{-1}{2a\sqrt{a}} = \dfrac{-1}{2\ a^{3/2}}}$

(II) $f'(5) = 2000$ means: When the interest rate is 5%, the instantaneous rate of change of repayment cost with respect to interest rate, is \$2000 per percent.

Grading: Total points possible = 6.

1 pt - Any reasonable attempt.

3.5 pt for (I): 0.5 pt = know/show a correct formula for $f'(a)$.
0.5 pt = correctly plug the given $f(x)$ into that formula.
2 pt = correct algebra, up to $\frac{-1}{x \cdot a \cdot (1/\sqrt{x} + 1/\sqrt{a})}$.
0.5 pt = plug in $(x = a)$ and get correct answer.

1.5 pt for (II): 1 pt = correct interpretation of $f'(5) = 2000$.
0.5 pt = include correct units in discussion.