Quiz 2 - 2/15/2022

(I) Sketch the graph of a function f that satisfies all the following requirements:

$$\lim_{x \to -2^-} f(x) = 1, \quad \lim_{x \to -2^+} f(x) = -3, \quad f(-2) = 0$$

$$\lim_{x \to 3} f(x) = 3, \quad f(3) = \text{undefined}$$

As always, graph must show all labels and symbols needed to read it correctly.

(II) Find a formula for the inverse of the function $f(x) = \dfrac{3 - 2x}{2 - x}$.   Show steps.

Solution

(I) At $x = -2$ the graph must break, to accommodate unequal limits on the left and right. Since $f(-2) = 0$, there is a closed circle on the x-axis at that point. At $x = 3$ there is a hole in the graph, because the limit exists, but the function is undefined. There are many possible correct solutions to this problem. One example of a function that satisfies all the requirements is shown below

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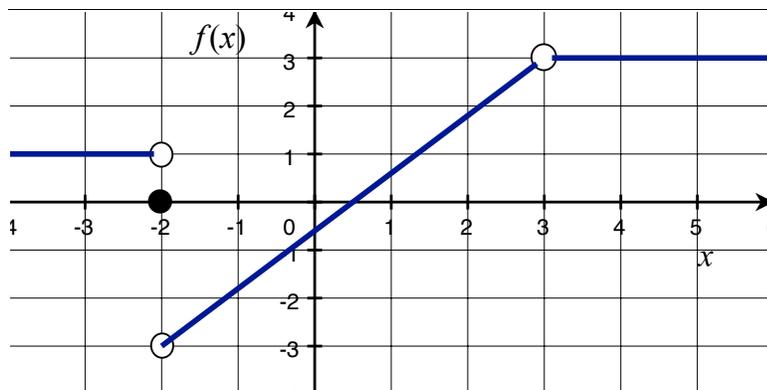
(II) To find the inverse, solve for x in terms of y.

The given equation is: $y = \dfrac{3 - 2x}{2 - x}$

Multiply both sides by $2 - x$: $y(2 - x) = 3 - 2x \implies 2y - xy = 3 - 2x$

Group all x-terms on one side: $2x - xy = 3 - 2y$

Factor out x and divide through by factor: $x(2 - y) = 3 - 2y \implies x = \dfrac{3 - 2y}{2 - y}$

Wow! Inverse is the same as the original function!

Answer: $\boxed{f^{-1}(x) = \dfrac{3 - 2x}{2 - x}}$

Verify answer (not a required part of the solution):

$$f(f^{-1}(x)) = \frac{3 - 2f^{-1}(x)}{2 - f^{-1}(x)} = \frac{3 - 2(3 - 2x)/(2 - x)}{2 - (3 - 2x)/(2 - x)} = \frac{3(2 - x) - 2(3 - 2x)}{2(2 - x) - (3 - 2x)} = \frac{x}{1} = x$$

Grading: Total points possible $= 6$.

3 pt for (I):   0.5pt for each of the following 6 features:
        (a) correct left-limit at $x = -2$, including open circle
        (b) correct right-limit at $x = -2$, including open circle
        (c) correct value of $f(-2)$, shown with dot or with closed circle
        (d) correct limit from left & right at $x = 3$
        (e) correctly leave $f(3)$ undefined
        (f) graph shows all needed axes labels

3 pt for (II):   0.5pt $=$ Attempt to flip x, y, (or otherwise) solve for y.
        2pt $=$ correct algebraic steps till getting $x = \frac{3-2y}{2-y}$.
        0.5pt $=$ correctly express final result in the form $y = \cdots$, or $f^{-1}(x) = \cdots$.