Woksheet on limits, continuity, and rates of change

1. Find the indicated limits using algebraic methods:

   (a) $\displaystyle\lim_{x\to1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$

   (b) $\displaystyle\lim_{x\to-1}\frac{x^3+x^2-x-1}{x+1}$

   (c) $\displaystyle\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}$

   (d) $\displaystyle\lim_{x\to0}\frac{\sqrt{x}}{\sqrt{1+\sqrt{x}}-1}$

   (e) $\displaystyle\lim_{x\to 2}\left(\frac{x^2}{4-x^2}+\frac{1}{x-2}\right)$

   (f) $\displaystyle\lim_{x\to-2}f(x)$ and $\displaystyle\lim_{x\to2}f(x)$

   for the function $f(x)=\dfrac{|x^2-4|}{x+2}$

   (g) Find $\displaystyle\lim_{x\to-2}2x+|x+2|$

   (h) $\displaystyle\lim_{x\to1}\frac{5x^2-5}{|x-1|}$

2. The graphs of two functions f and g are given below. Use them to evaluate each of the following limits (give a mathematical justification for your answers):

   (a) $\displaystyle\lim_{x\to-1}[f(x)+g(x)]$

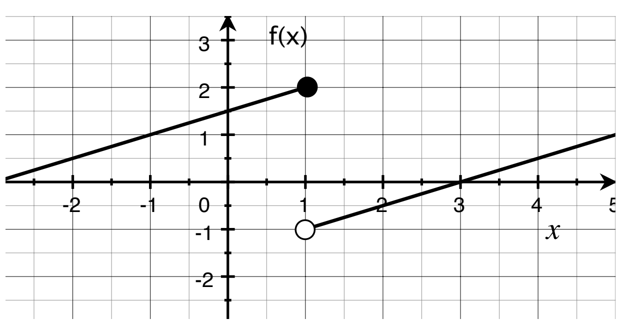
   (b) $\displaystyle\lim_{x\to-1}[f(x)\cdot g(x)]$

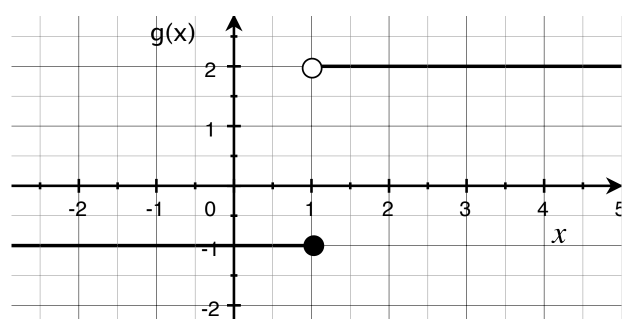
   (c) $\displaystyle\lim_{x\to1}[f(x)+g(x)]$

   (d) $\displaystyle\lim_{x\to1}[f(x)\cdot g(x)]$

   (e) $\displaystyle\lim_{x\to0}\frac{f(x)}{g(x)}$

   (f) $\displaystyle\lim_{x\to0}\frac{f(x)+2}{g(x)}$





3. Consider the function

$$f(x)=\begin{cases}\dfrac{6}{x-4}, & \text{if } x<-2\\ 5-3x, & \text{if } -2\le\ x\le1\\ \dfrac{x-1}{\sqrt{x}-1}, & \text{if } x>1\end{cases}$$

   Find the following limits: (a) $\displaystyle\lim_{x\to-2}f(x)$, (b) $\displaystyle\lim_{x\to1}f(x)$.

4. Given the same function as in the previous exercise, find all the values of x for which f is continuous. Use the mathematical definition of continuity to justify all claims.

5. Determine all the values of a that would make the following function continuous for all x

$$f(x) = \begin{cases} 2, & \text{if } x < 3 \\ a^2x + a, & \text{if } x \geq 3 \end{cases}$$

6. Find: (a) $\lim\limits_{x \to 0} \; x^4 \sin\left(\frac{1}{x}\right)$ (b) $\lim\limits_{x \to 0} \; x^2 \, e^{\cos\left(\frac{1}{x}\right)}$

7. In the year 2011, the United States spent about \$800 billion on its military and warfare operations (higher than the military spending of the next 17 countries combined). The following function, based on actual data from the U.S. government, is a model for military spending in the U.S. since the year 2000 (t denotes the number of years after 2000, and f the military spending in billions of dollars)

$$f(t) = \frac{300}{0.3 + 0.7\, e^{-0.2t}}$$

(a) Find the average change in military spending over the time period beginning with the year 2011 and lasting: 2 years, 1 year, and 0.5 year. In other words, we want to compute three separate averages here.
(b) From these averages, estimate the slope of the tangent line to the graph of $f(t)$ when the year is 2011. Find the equation of the tangent line.

8. Use the Intermediate Value Theorem to show the following equations have a root. Be sure to check that the theorem's hypotheses are satisfied before applying it.

(a) $x^2 + \sin(\pi x) = 3$ (b) $x \ln(x+1) = 4x - 2$

Solution outline/grading notes

1. Find the indicated limits using algebraic methods:

   (a) $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$:

   Common denominator & reduce to $\lim_{x\to 1}\left(\frac{1}{x+1}\right)$ $\boxed{=1/2}$

   (b) $\lim_{x\to -1}\frac{x^3+x^2-x-1}{x+1}$:

   Factor & cancel to get $\lim_{x\to -1}(x-1)(x+1)$ $\boxed{=0}$

   (c) $\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}$:

   Use $(3+\sqrt{x})$ to rationalize & simplify to $\lim_{x\to 9}\frac{-1}{(9+x)(3+\sqrt{x})}$ $\boxed{=-\frac{1}{108}}$

   (d) $\lim_{x\to 0}\frac{\sqrt{x}}{\sqrt{1+\sqrt{x}}-1}$:

   Use $(\sqrt{1+\sqrt{x}}+1)$ to rationalize & reduce to $\lim_{x\to 0}(\sqrt{1+\sqrt{x}}+1)$ $\boxed{=2}$

   (e) $\lim_{x\to 2}\left(\frac{x^2}{4-x^2}+\frac{1}{x-2}\right)$:

   Common denominator, factor & simplify to $\lim_{x\to 2}\left(\frac{-(x+1)}{x+2}\right)$ $\boxed{=-3/4}$

   (f) $\lim_{x\to -2}f(x)$ and $\lim_{x\to 2}f(x)$ for the function $f(x)=\frac{|x^2-4|}{x+2}$:

   Take left/right limits & get $\lim_{x\to -2^-}f(x)=-4$, $\lim_{x\to -2^+}f(x)=4$, $\lim_{x\to -2}f(x)=\text{DNE}$

   For $x\to 2$, $\lim_{x\to 2^-}f(x)=0$, $\lim_{x\to 2^+}f(x)=0$. Thus, $\lim_{x\to 2}f(x)=0$.

   (g) Find $\lim_{x\to -2}2x+|x+2|$:

   Take left/right limits & get $\lim_{x\to -2^-}f(x)=-4$, $\lim_{x\to -2^+}f(x)=-4$, $\lim_{x\to -2}f(x)=-4$

   Answer $\boxed{=-4}$

   (h) $\lim_{x\to 1}\frac{5x^2-5}{|x-1|}$:

   Take left/right limits & get $\lim_{x\to 1^-}\frac{5(x^2-1)}{-(x-1)}=\lim_{x\to 1^-}-5(x+1)=-10$

   $\lim_{x\to 1^+}\frac{5(x^2-1)}{(x-1)}=\lim_{x\to 1^+}5(x+1)=10$. Therefore, $\lim_{x\to 1}\frac{5x^2-5}{|x-1|}=\boxed{\text{DNE}}$

2. Answers only:

(a) $\lim\limits_{x\to -1}[f(x)+g(x)] = 0$

(b) $\lim\limits_{x\to -1}[f(x)\cdot g(x)] = -1$

(c) $\lim\limits_{x\to 1}[f(x)+g(x)] = 1$

(d) $\lim\limits_{x\to 1}[f(x)\cdot g(x)] = -2$

(e) $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = -1.5$

(f) $\lim\limits_{x\to 0}\dfrac{f(x)+2}{g(x)} = -3.5$

3. Must consider left/right limits for each of the indicated problems. Answers:

(a) $\lim\limits_{x\to -2^-} f(x) = -1, \quad \lim\limits_{x\to -2^+} f(x) = 11, \quad \lim\limits_{x\to -2} f(x) = \text{DNE}$

(b) $\lim\limits_{x\to 1^-} f(x) = 2, \quad \lim\limits_{x\to 1^+} f(x) = 2, \quad \lim\limits_{x\to 1} f(x) = 2$

4. f is continuous on the intervals: $x < -2, \quad -2 < x < 1, \quad x > 1$ by continuity theorems. f is discontinuous at $x = -2$ because its limit is undefined there.
f is continuous at $x = 1$ because $\lim\limits_{x\to 1} f(x) = f(1) = 2$.
So, the answer is: f is continuous for all x except $x = -2$.

5. The function will be continuous at $x = 3$ if $\lim\limits_{x\to 3} f(x) = f(3)$. In order to make that happen we must have: $2 = 3a^2 + a$.
Thus, $a = 2/3$ and $a = -1$ will both work to make f continuous for all x.

6. (a) $\lim\limits_{x\to 0} \; x^4 \sin\left(\dfrac{1}{x}\right)$

First, show that $-x^4 \le x^4 \sin\left(\dfrac{1}{x}\right) \le x^4$ for all x.

Then use the squeeze theorem and get $\lim\limits_{x\to 0} \; x^4 \sin\left(\dfrac{1}{x}\right) = 0$

(b) $\lim\limits_{x\to 0} \; x^2\, e^{\cos\left(\frac{1}{x}\right)}$

Show that $\dfrac{x^2}{e} \le x^2\, e^{\cos\left(\frac{1}{x}\right)} \le x^2 e$ for all x.

Then use the squeeze theorem and get $\lim\limits_{x\to 0} \; x^2\, e^{\cos\left(\frac{1}{x}\right)} = 0$