Homework Solutions for Sec. 4.3 (2nd part), 4.5, 4.6

Assigned exercises: Sec. 4.3: 51, 64, 65.
Sec. 4.5: 7, 11, 13, 19. 24, 28, 30, 33, 73
Sec. 4.6: 4, 5, 11, 13, 32a, 38.

Graded exercises circled.
Grading scheme: 3 points each for 5 graded problems, plus 10 points for completing the rest. Take off 0.5 point for each missing prob.

Sec. 4.3: Exercise 64

$f'(x)$ exists for all x, according to the information given.
So, we can apply the Mean Value Theorem on the interval $[2,8]$.
By the M.V.T., there exists some c between $(2,8)$ such that

$$\frac{f(8)-f(2)}{8-2} = f'(c) \implies f(8)-f(2) = 6f'(c)$$

But, the given info. also suggests $3 \le f'(c) \le 5$

$$\implies 3\times 6 \le 6f'(c) \le 5\times 6$$

It follows that $3\times 6 \le f(8)-f(2) \le 5\times 6 \implies \boxed{18 \le f(8)-f(2) \le 30}$

Grade: 1 pt = mention somewhere that f' exists everywhere, or on $[2,8]$
1 pt = mention somewhere that by the MVT, $f'(c) = \frac{f(8)-f(2)}{6}$
1 pt = use the inequality for $3 \le f'(c) \le 5$ to get final result.

Sec. 4.5: Exercise 11

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$

Check whether indeterminate: $\frac{\ln\infty}{\sqrt{\infty}} \sim \frac{\infty}{\infty} \Rightarrow$ indeterminate quotient.

Apply L.H. Rule: $\lim_{x\to\infty} \frac{(\ln x)'}{(x^{1/2})'} = \lim_{x\to\infty} \left[\frac{1/x}{(\frac{1}{2}x^{-1/2})} = \frac{2x^{1/2}}{x} = \frac{2}{x^{1/2}}\right]$

$\lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$, because it is of the form $\lim_{x\to\infty} \frac{1}{x^n}$

Answer: $\boxed{\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0}$

Grade: 0.5 pt = check for indeterminate & conclude it is.
1 pt = correctly differentiate numerator & denom for L.H. rule
1 pt = simplify it correctly; 0.5 pt = get final answer.

Sec. 4.5: Exercise 24

$$\lim_{x\to 0} \frac{e^{x}-e^{-x}-2x}{x-\sin(x)}$$ check $\rightarrow \frac{e^0-e^0-0}{0-\sin 0} \sim \frac{0}{0}$ Indeterminate.

Apply L.H. rule: $\lim_{x\to 0} \frac{(e^x-e^{-x}-2x)'}{(x-\sin x)'} = \lim_{x\to 0} \frac{e^x+e^{-x}-2}{1-\cos x}$

Plug in $x=0$ and check: $\frac{e^0+e^0-2}{1-1} \sim \frac{0}{0}$. It is still indeterminate.
Apply L.H. rule again:

$\lim_{x\to 0} \frac{(e^x+e^{-x}-2)'}{(1-\cos x)'} = \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} \sim \frac{1-1}{0} = \frac{0}{0}$. still indet!

Apply L.H. once again: $\lim_{x\to 0} \frac{(e^x - e^{-x})'}{(\sin x)'} = \lim_{x\to 0} \frac{e^x+e^{-x}}{\cos(x)} = \frac{1+1}{1} \boxed{=2}$

$\boxed{\text{Answer: } 2}$

Grade: 1 pt each for the first 2 rounds of applying L.H. rule.
(0.5pt. for checking indeterminate, 0.5pt for differentiating)
1 pt for last round & figuring out the answer.

Sec. 4.6: Exercise 4

The sum of 2 positive #s is 16. What is the smallest possible sum of their squares.

Let x, y be the 2 positive #s.

Then, $x+y=16$, and the sum of their squares $= x^2+y^2$

Let $f(x) = x^2 + (16-x)^2$, where we have used $y = 16-x$

We want to find abs. minimum of $f(x)$.

$f'(x) = 2x + 2(16-x)(-1)$. Then $f'=0 \Rightarrow 2x - 2(16-x) = 0$

$\Rightarrow x - 16 + x = 0 \Rightarrow \boxed{x=8}$

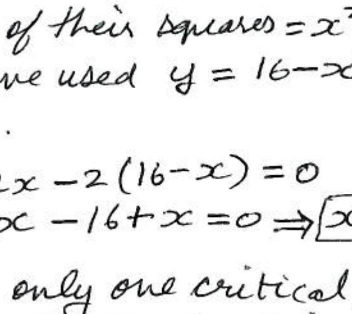
$f'' = 2+2 = 4$.

Since we have a continuous function with only one critical point, and $f'' > 0$, the C.P. gives both the local & the abs. minimum

Answer: $\boxed{\text{The 2 numbers are 8 and 8.}}$

Grade: 0.5 pt = introduce 2 variables and get $x+y=16$
1 pt = figure out the function $f(x) = x^2+(16-x)^2$
0.5pt = find the critical point at $x=8$
0.5pt = explain/justify why it gives abs. minimum
0.5 pt = correctly give answer saying the 2 numbers are 8.

Sec. 4.6: Exercise 38



Suppose she rows the boat at angle θ, as shown, to the point B. After that she gets out & walks to C.

Since angle $BAO = \theta$, angle $BOC = 2\theta$ (Why?)

Distance $AB = AC \cdot \cos(\theta) = 4\cos(\theta)$ mi [Given radius = 2]

Time to row this distance $= \frac{4\cos(\theta)}{2} = 2\cdot\cos(\theta)$ hours [Because speed of rowing $= 2 \frac{mi}{hr}$]

Distance along arc $BC = 2\theta \cdot r = 4\theta$ mi

Time to walk this distance $= \frac{4\theta}{4} = \theta$ hours [$\because$ walking speed $= 4 \frac{mi}{hr}$]

Total time taken is: $\boxed{f(\theta) = 2\cdot\cos(\theta) + \theta}$ hours

Minimize: $f'(\theta) = -2\cdot\sin(\theta) + 1$. Thus, $f'=0 \Rightarrow \sin\theta = \frac{1}{2}$
$\Rightarrow \theta = 30°$ or $\frac{\pi}{6}$

Since $f(\theta)$ is a function on the domain $\theta = 0$ to $\frac{\pi}{2}$, we can find abs. minimum by plugging in C.P. and end points.

The abs minimum is at $\pi/2$ and $f(\pi/2) = \pi/2$ hours.

θ	$f(\theta)$
0	2
$\pi/6$	$\sqrt{3} + \pi/6$
$\pi/2$	$\pi/2$

Answer: She should walk the entire distance.

Grade: 1.5pt = reasonable work leading to $f(\theta) = 2\cos\theta + \theta$
1.5 pt = reasonable steps leading to optimization & final answer.