Instructions:
- Answer all questions on separate paper (not on this sheet!).
- This part is a regular “closed-book” test, and is to be taken without the use of notes, books, calculator, or other reference materials.
- This test contains questions numbered (1) to (5).

It adds up to 20 points - 4 points for each question.

(1) For each differential equation below, state its order and classify it as linear or nonlinear.
(a) \( y''' + 2y' - 8y^2 = 0 \)
(b) \( e^{2t}y'' - y' + \sin(t) y = \cos(2t) \)
(c) \( y' + \frac{1}{y} = t \)
(d) \( (y')^2 + 4y = \sin t \)

(2) Determine whether the following differential equations are exact (show steps):
(a) \( (xe^x - ye^y) \frac{dy}{dx} = e^x(x + 3) \).
(b) \( (\sin y + 2xe^y)dx + (\sin x + x^2e^y - 1)dy = 0 \).

(3) (a) Show that \( \phi(t) = e^{2t} \) is a solution of \( y' - 2y = 0 \) and that \( y(t) = c\phi(t) \) is also a solution of this equation for any value of the constant \( c \).
(b) Show that \( \phi(t) = 1/t \) is a solution of \( y' + y^2 = 0 \) for \( t > 0 \), but that \( y(t) = c\phi(t) \) is not a solution of this equation except if \( c = 0 \) or \( c = 1 \).
(c) Comment on why \( y(t) = c\phi(t) \) works for problem (a) but not for (b).

(4) Sometimes it is possible to solve a nonlinear differential equation by making a change of the dependent variable to convert it into a linear equation. One important 1st order DE of this type is the Bernoulli equation, which has the form: \( y' + p(t)y = q(t)y^n \).
Show that this equation can be made linear in the variable \( u = y^{1-n} \) by multiplying the original DE by \( (1 - n)y^{-n} \).

(5) Suppose a population \( P(t) \) satisfies the logistic equation
\[
\frac{dP}{dt} = rP \left( 1 - \frac{P}{K} \right)
\]
where \( r \) is the intrinsic growth rate and \( K \) the carrying capacity. Show that the population grows fastest when it is at half the carrying capacity. For full credit, solution logic and reasoning must be as clear and correct as any computational details.

End of test
Earlham College
MATH 320 : Differential Equations : Fall 2013

Take home portion of Test 1 - administered via the Science Library. Test must be taken between September 23-25 (by 10 AM). Complete and return to Science Library within 24 hours after checkout, or before library closing time, whichever comes first.

Instructions:
• Answer all questions on separate paper – not on this sheet!
• You may use the following reference materials: The textbook, your own class notes and homework, and a calculator.
• Prohibited materials: Any other reference sources, including electronic, printed, written or verbal.
• For full credit, all integrations must be done by hand, with clear, detailed steps.
• This test consists of questions numbered (1) to (4). Each is worth 5 points. Total points = 20.

(1) Find the general solution of: 
\[ (2y - xy \ln x)dx - 2x \ln x \, dy = 0. \]

(2) Solve the initial value problem
\[ (x^2 - 1)y' + 2xy - \cos x = 0, \quad y(0) = 1 \]
and find the largest interval in which your solution holds.

(3) Show that if \( a \) and \( \lambda \) are positive constants, and \( b \) is any real number, then every solution of the differential equation \( y' + ay = be^{-\lambda t} \) has the property that \( y \to 0 \) as \( t \to \infty \). (Hint: Consider the cases \( a = \lambda \) and \( a \neq \lambda \) separately.)

(4) Let \( \frac{dy}{dt} = f(y) \) be an autonomous differential equation, with \( f(y) \) shown below

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
 y & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
 f(y) & & & & & & & & & & & \\
\end{array}
\]

(a) Find the equilibrium solutions of the differential equation, determine their stability (i.e., stable, semi-stable, unstable) and sketch the phase line. Show reasoning.

(b) Sketch qualititatively reasonable graphs of the solution behavior for the following 4 initial conditions: \( y(0) = -3, \ y(0) = 0, \ y(0) = 1, \ y(0) = 2. \)

End of test
Differential Equations: Fall 2013: Test 1 Solutions

In-class Part

[1] (a) \( y'' + 2y' - 8y^2 = 0 \) is 3rd order, non-linear (\( \cdot 8y^2 \) term)
(b) \( e^{2t} y'' - y' + \sin(t) y = \cos(2t) \) is 2nd order, linear
(c) \( y' + \frac{1}{y} = t \) is 1st order, non-linear (\( \cdot \frac{1}{y} \) term)
(d) \( (y')^2 + 4y = \sin t \) is 1st order, non-linear (\( \cdot (y')^2 \) term)

[2] For exactness, \( M_y = N_x \) for DE of the form \( M(x,y) \frac{dy}{dx} + N(x,y) \frac{dx}{dy} = 0 \)
(a) \( (xe^{-y} - ye^{-y}) \frac{dy}{dx} = e^{-y}(x+3) \Rightarrow e^{-y}(x+3) \frac{dy}{dx} - (xe^{-y} - ye^{-y}) = 0 \)

For \( M_y \) we have: \( \frac{\partial}{\partial y} [e^{-y}(x+3)] = -xe^{-y} \)
For \( N_x \) we have: \( \frac{\partial}{\partial x} [-xe^{-y} + ye^{-y}] = -xe^{-y} - e^{-y} \)
Since \( M_y \neq N_x \), this DE is not exact.

(b) \( (\sin y + 2xe^{y}) \frac{dx}{dy} + (\sin x + x^2 e^{y} - 1) \frac{dy}{dx} = 0 \)
\( M = \sin y + 2xe^{y} \Rightarrow \frac{\partial M}{\partial y} = \cos y + 2xe^{y} \)
\( N = \sin x + x^2 e^{y} - 1 \Rightarrow \frac{\partial N}{\partial x} = \cos x + 2xe^{y} \)
Again, since \( M_y \neq N_x \), the given DE is not exact.

[3] (a) Given DE is: \( y' - 2y = 0 \)
Plug in \( y = \phi(t) = e^{2t} \): \( y' - 2y = 2e^{2t} - 2(e^{2t}) = 0 \)
Therefore, \( y = e^{2t} \) is a solution.
Plug in \( y = c \cdot e^{2t} \): \( y' - 2y = 2ce^{2t} - 2(c \cdot e^{2t}) = 0 \)
Thus, \( y = c \cdot e^{2t} \) is a solution for any constant \( c \).

[b] Given DE is: \( y' + y^2 = 0 \)
Plug in \( y = 1/t \) and check: \( \left( \frac{1}{t} \right)' + \left( \frac{1}{t} \right)^2 = -\frac{1}{t^2} + \frac{1}{t^2} = 0 \)
Therefore, \( y = 1/t \) is a solution.
Plug in \( y = c \cdot t \) and check: \( \left( \frac{c}{t} \right)' + \left( \frac{c}{t} \right)^2 = -\frac{c}{t^2} + \frac{c^2}{t^2} \neq 0 \)
So, \( y = c \cdot t \) is not a solution unless \( c = 0, 1 \) or
except for select values of \( c \).

[c] The problem in (a) is linear, while the one in (b) is non-linear.
Also, the problem (a) is homogeneous. Therefore, if \( y = \phi(t) \) is a solution, \( y = c \cdot \phi(t) \) would be a solution for the linear homogeneous problem, but not necessarily for the non-linear one.
[4] The given equation has the form: \( y' + p(t) \frac{y}{y} = q(t) \frac{y}{y} \)

As instructed, if we multiply through by \((1-n)\frac{y}{y} \), we get

\[
(1-n)\frac{y}{y} y' + (1-n)\frac{y}{y} p(t) y = (1-n)\frac{y}{y} q(t) y
\]

\[
\Rightarrow \left[y^{(1-n)}\right]' + (1-n)p(t) y^{(1-n)} = (1-n)q(t)
\]

\[
\Rightarrow \quad \frac{u'}{(1-n)p(t) u} = (1-n)q(t), \quad \text{with} \quad u = y^{(1-n)}
\]

[5] The population grows fastest when \( \frac{dP}{dt} \) is maximized.

There are 2 approaches here: Maximize \( \frac{dP}{dt} \) with respect to \( t \), or with respect to \( P \). Both lead to the same answer, but it is probably more correct, technically, to differentiate and maximize with respect to \( t \).

Given \( \frac{dP}{dt} = rP(1 - \frac{P}{K}) \), where \( r, K \) are constants,

So, \( \frac{d}{dt} \left[ \frac{dP}{dt} \right] = rP' \left(1 - \frac{P}{K}\right) + rP \left(0 - \frac{P'}{K}\right) = rP' - \frac{rPP'}{K} - \frac{rPP'}{K} \)

\[
\Rightarrow rP' \left[1 - \frac{2P}{K}\right] = rP \left(1 - \frac{P}{K}\right) \left[1 - \frac{2P}{K}\right]
\]

Setting \( \frac{d}{dt} \left[ \frac{dP}{dt} \right] = 0 \), we get: \( P = 0, P = K \) and \( P = \frac{K}{2} \)

The sign graph of \( \frac{d}{dt} \left[ \frac{dP}{dt} \right] \) is shown below:

\[
\begin{array}{ccc}
- & + & - \\
0 & \frac{K}{2} & K
\end{array}
\]

This shows that \( P = \frac{K}{2} \) corresponds to a maximum of \( \frac{dP}{dt} \). Therefore, the population grows fastest when it reaches half the carrying capacity.

Grading Notes
[1] (a) = (b) = (c) = (d) = 1 point
[2] (a) = (b) = 2 points
0.5 → Know \( M_y = Nx \) condition
1.5 → Apply it correctly & get answer
[3] (a) = 1 pt, (b) = 2 pt, (c) = 1 pt.

![Grading Notes]

[4] 1 pt → Multiply through by \((1-n)\frac{y}{y} \)
3 pt → figure out how to complete the problem
2.5 → get correct steps to \( P = K/2 \)
0.5 → justifying why maximum
Differential Equations: Fall 2013: Test 1 Solutions

Take-home Part

[1] Find general solution: \((2y-x \cdot \ln x) \, dx - 2x \cdot \ln x \, dy = 0\)

Rearrange and see whether it is separable:

\[y \cdot (2-x \cdot \ln x) \, dx = 2x \cdot \ln x \, dy\]

\[\Rightarrow \frac{(2-x \cdot \ln x) \, dx}{2 \cdot x \cdot \ln x} = \frac{dy}{y} \quad \Rightarrow \text{Separable!}\]

\[\left(\frac{1}{x \cdot \ln x} - \frac{1}{2}\right) \, dx = \frac{dy}{y} \Rightarrow \text{Integrate both sides}\]

\[\int \frac{dx}{x \cdot \ln x} - \frac{1}{2} \int dx = \int \frac{dy}{y} \Rightarrow \ln(\ln x) - \frac{x}{2} = \ln y + c_1\]

Rearrange/solve for \(y\):

\[\ln y = c_2 + \ln(\ln x) - \frac{x}{2}\]

\[\Rightarrow y = e^{c_2 + \ln(\ln x) - \frac{x}{2}} = c \cdot \ln(\ln x) - e^{-\frac{x}{2}}\]

\[\Rightarrow y = c \cdot \ln x \cdot e^{-\frac{x}{2}}\]

\(c = \text{arbitrary constant}\)

[2] Solve the I.V.P. \((x^2 - 1) \, y' + 2xy - \cos x = 0\), \(y(0) = 1\)

Rearrange:

\[y' + \frac{2x}{x^2 - 1} \cdot y = \frac{\cos x}{x^2 - 1}, \quad x \neq \pm 1\]

Doesn't look separable in any obvious way. So, either try integrating factor for 1st-order linear case, or try to see if it is exact.

\[\frac{dy}{dx} = \frac{\cos x - 2xy}{x^2 - 1} \Rightarrow (x^2 - 1) \, dy = (\cos x - 2xy) \, dx\]

\[\frac{\partial M}{\partial y} = -2x \quad \frac{\partial N}{\partial x} = -2x\]

\(\Rightarrow \text{it is exact!}\)

So, we must find \(f(x,y)\) such that

\[df = (\cos x - 2xy) \, dx - (x^2 - 1) \, dy\]

\[\Rightarrow \frac{\partial f}{\partial y} = -(x^2 - 1) \quad \text{so, } f(x,y) = -(x^2 - 1) \, y + g(x)\]

Differentiate with respect to \(x\):

\[\frac{\partial f}{\partial x} = -2xy + g'(x)\]

Equate to \(M(x,y)\):

\[-2xy + g'(x) = \cos x - 2xy\]

\[g'(x) = \cos x \Rightarrow g(x) = \sin x\]
Therefore, \( f(x,y) = -(x^2) y + \sin x = y(1-x^2) + \sin x \)

The general solution is then:

\[
y(1-x^2) + \sin x = C
\]

or:

\[
y = \frac{C - \sin x}{1-x^2}, \quad C = \text{arbitrary constant}
\]

Plug in initial condition: \( y(0) = 1 \Rightarrow 1 = \frac{C - 0}{1-0} \)

Largest interval:

\[
y = \frac{1 - \sin x}{1-x^2}, \quad -1 < x < 1
\]

[3] \( y' + ay = be^{-\lambda t} \)

want to show that \( y \to 0 \) as \( t \to \infty \).

First, suppose we consider the case where \( a = \lambda \).

Then, we have:

\[
y' + ay = be^{-\lambda t}
\]

An integrating factor for this 1st order linear ODE is \( \mu = e^{\lambda t} \).

Multiply by \( \mu \) and simplify:

\[
(ye^{\lambda t})' = be^{-\lambda t} \cdot e^{\lambda t} \Rightarrow ye^{\lambda t} = \int b dt = bt + c
\]

\[
y = (bt + c)e^{-\lambda t}, \quad \text{for} \quad c = \text{arbitrary constant}
\]

Next, suppose \( a \neq \lambda \). Then the ODE becomes:

\[
y' + ay = be^{-\lambda t} \quad \text{we still have} \quad \mu = e^{\lambda t}, \quad \text{and the result is}
\]

\[
(ye^{\lambda t})' = be^{-\lambda t} \cdot e^{\lambda t} \Rightarrow ye^{\lambda t} = \int b e^{(a-\lambda)t} dt
\]

So,

\[
y e^{\lambda t} = b \cdot \frac{e^{(a-\lambda)t}}{a-\lambda} + c
\]

\[
y = \frac{b}{a-\lambda} e^{-\lambda t} + ce^{-\lambda t}
\]

Putting both results together we have:

\[
y = \begin{cases} 
(bt + c)e^{-\lambda t}, & \text{if } a = \lambda \\
\frac{b}{a-\lambda} e^{-\lambda t} + ce^{-\lambda t}, & \text{if } a \neq \lambda 
\end{cases}
\]

Since \( a \) and \( \lambda \) are positive, we can see that \( y \to 0 \) as \( t \to \infty \)

in both cases, because \( e^{-\lambda t} \to 0 \), \( e^{-at} \to 0 \) and \( t e^{-at} \to 0 \).

[4] (a) From the graph we see that \( \frac{dy}{dt} = 0 \) when \( f(y) = 0 \), which

happens at \( y = -4, -2, 1 \) and 4. Thus these 4 \( y \)-values

are equilibrium solutions (or, equilibrium points).
The stability of the equilibrium points is determined by the sign of dy/dt before and after each point. This is easiest to see from a sketch of the phase line.

\[ y = -4 \text{ is stable} \]
\[ y = -2 \text{ is unstable} \]
\[ y = 1 \text{ is semi-stable} \]
\[ y = 4 \text{ is stable} \]

(b)

From the stability characteristics of the equilibrium solutions, we can deduce the qualitative behavior of solutions that start from different initial conditions.

The graphs corresponding to \( y(0) = -4 \), \( y(0) = 0 \), \( y(0) = 1 \) and \( y(0) = 2 \) are shown here.

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**Grading Notes**

1. **1 pt → find a workable strategy**, 3 pt → implement it correctly, 1 pt → express general solution correctly & explicitly
2. **1 pt → find workable strategy for gen. solution**, 3 pt → implement correctly, 1 pt → plug in initial cond. & find particular solution
3. **2 pt + 2 pt → solve for 2 cases**: \( a = \lambda \) and \( a \neq \lambda \)
   1 pt → explain why \( y \to 0 \) as \( t \to \infty \)
4. \( (a) = 3 \) points, \( (b) = 2 \) points
   (a) 1 pt each → correct equilibrium solution; correct stability; correct phase line