Chapter 5: Market Equilibrium Theories

The mean-variance analysis in combination with the separation theorem provides a powerful tool for the investor. According to mean-variance analysis, if the investor wishes to maximize the expected return on a portfolio – while minimizing risk – then they should determine the expected rate of return and risk of the individual assets, construct the efficient frontier, construct the capital allocation line to find the optimum risky portfolio in combination with the risk-free asset, then find the portfolio along line where they feel comfortable with the level of risk. Clearly, this is a fairly daunting task for an individual investor, but not necessarily for an investment firm. The investment firm will employ people specially trained to do the security analysis required to arrive at the best estimates of expected return and risk. In addition, the firm will have computer specialists able to minimize the computing time required to construct the efficient frontier and capital allocation line. The discovery of the optimum risky portfolio will require adjustments from time-to-time, which the firm may want to minimize to reduce transaction costs. The final step is to discover the client’s tolerance for risk and any special constraints (e.g., tax issues, environmental and social concerns, etc.). There are fairly standard questionnaires that will help the salesperson do this now. Thus far, we have a pretty good approach to running an investment firm.

There are two broad problems with the mean-variance analysis. First, from an applied perspective, the procedure described above is still a bit much. It would be nice to find a method of simplifying things. Second, from an academic perspective, the mean-variance analysis by itself does not provide an explanation of equilibrium prices. In this regard, the theory is prescriptive in the sense of telling what the investor should do in order to maximize return and minimize risk. It is not a theory that sheds light on asset prices we observe. It, again, would be nice to extend this tool into a theory of asset prices. The Capital Asset Pricing Model (CAPM) derived in this chapter is intended to address both of these problems.

The chapter is laid out in the following sections. The next section reviews some basic microeconomic ideas, which will be useful to use as analogies. Section 5.2 briefly describes the market portfolio. Next, we take another look at the variance of a portfolio. The variance of an asset will be broken into two components – one of which will be able to be diversified away within a portfolio. Section 5.4 goes onto derive the CAPM. Section 5.5 illustrates some of the uses to which the CAPM has been put. Section 5.6 demonstrates how the CAPM gets operationalized in single factor models. Section 5.7 presents an alternative equilibrium theory – the Arbitrage Pricing Theory (APT). Although built from a different foundation, the APT will arrive at a similar place to CAPM.

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1 The firm may not want to do this, if they earn fees from trading. On the other hand, these fees are now widely reported so that the firm’s performance is adjusted to account for the fees.
5.1 Economic Concepts

The current section covers three economic concepts. The first, alluded to several times above, comes from consumer theory. The second reviews market equilibrium. The third is the law of one price. The first two will help to interpret the CAPM. The third is used in the construction of APT.

5.1.2 Consumer Theory

5.1.3 Market Equilibrium

5.1.4 Law of One Price

5.2 The Market Portfolio

The market portfolio is composed of all assets held in the same proportion as represented in the market as a whole. The proportions are defined in terms of the value rather than number of shares or any other physical units. Though unrealistic to be sure, the market portfolio plays an important role in the capital asset pricing model (CAPM). It is unrealistic for several reasons. First, some assets are simply not traded. Second, there are an overwhelming number of assets available to choose from in the actual economy. However, the theoretical construction of the market portfolio will be useful. In practice, there exist index funds that present good proxies to the market portfolio (e.g., S&P 500).

In order to illustrate the meaning of the market portfolio, we will look at an extremely simple economy. Suppose an economy was completely described by two identical twins (Ted and Ned), two risky assets, and one risk-free asset. The twins use the same information and expectations (i.e., homogenous expectations) regarding expected returns to perform a mean-variance analysis. Each will construct an identical efficient frontier and discover the optimum risky portfolio. We will allow the individuals to differ only in terms of their preferences for the risk-return tradeoff and their initial endowment of wealth. Hence, the twins will invest in the same – optimum – risky portfolio, but their complete portfolio (including the risk-free asset) will differ. The table summarizes the situation.
Ted begins with $300 to invest in his complete portfolio. He puts 10% (or, $30) into purchasing the risk-free asset (e.g., government T-bill). From the table, we see that the optimum risky portfolio is composed of 60% of asset 1 and 40% of asset 2. Ted invests the remaining $270 into these assets accordingly. Ned has started out with a higher initial wealth and begins by investing 20% into the risk-free asset, then proceeds to invest in the two risky-assets according to the same weights Ted had used.

The optimum portfolio of risky assets (i.e., assets 1 and 2) held by the twins is in fact the market portfolio.

Assuming a given number of shares for each asset, Table 2 demonstrates that the size of the market ($670) is composed of the exact proportions of assets held by the two individuals. In order to arrive at the asset price we have simply taken the total investment into the assets from each individual and divided by the number of shares. The end result is that each individual is holding the same risky portfolio and by simply summing their holdings we arrive at the market as a whole. Thus, under the assumptions in place, the optimum risky portfolio will turn out to be the market portfolio.

Figure 5.1 demonstrates the situation graphically. The return-variability ratio (RV) for the market portfolio can be written as follows.

\[
(5.1) \quad RV = \frac{E(R_M) - r_f}{\sigma_M}
\]

\(^2\) Notice the implications of assuming a given number of shares. This makes the supply of each asset completely vertical.
The return-variability ratio states the risk premium for the market as a whole. That is, it states that market return in excess of the risk-free return per unit of risk. Since all assets are held as part of the market portfolio, it would be very interesting to understand how each asset contributes to forming the market return-variability ratio. Before doing so, it is necessary to understand the components of an asset’s variability.

5.3 The Components of Variability

The actual – as opposed to the expected - rate of return of an individual asset or portfolio can be thought to be composed of three parts. Consider having invested in one particular stock, before the investment you had expected a rate of return of say 12%. However, the actual rate of return had turned out to be 10%. What might have caused the actual return to deviate from what you expected? It could be that something specific – and surprising, at least to you – occurred to the underlying physical asset. That is, the corporation could have announced that the CEO left for another job. It could be that the industry this corporation operated in had an unforeseen downturn in demand or some sort of cost shock. In other words, many things could happen to the corporation or its industry which come as a surprise to investors and cause the return to deviate from expectations. Since these things come as a surprise we say that they are random and the contribution of these specific things to the actual return is a random variable.

The stock in our example does not exist in isolation. It is part of a larger market and could be influenced by movements in the market as a whole. If, for example, the NYSE plunges – stock prices falling – until 20% of the value of the market is destroyed, then the individual stock could certainly be impacted. The individual stock price might go down with the market – falling more or less than 20%. In any case, the actual rate of return of the stock could deviate from the expected due to unexpected changes in the market as a whole.

The actual rate of return of an individual asset is composed of the expected rate of return, relation to the market return, and unexpected events impacting the individual stock (or, the industry in which the issuer of the stock operates).

\[ r_{actual} = E(r) + \beta r_M + e \]

The first term on the right hand side is the expected rate of return on the asset. The second term describes how the actual return will be affected by the market return. The beta (\( \beta \)) term describes how much the actual return will change when the market return changes. The final term (\( e \)) is the random variable describing the impact of unexpected changes in factors specific to the issuer of the stock or its industry. In our example, the actual return was 2% below the expected. How do we explain this? It could be that the market return fell be 1% and this stock moves twice as much as the market – i.e., \( \beta = 2 \). Alternatively, it could be that an adverse surprise – e.g., CEO resigns – caused the return to drop below expected.
We need not dwell on equation (5.2); our purpose is to use it to study variability of the return on an individual asset. To do this, we state the variance of the rate of return on an asset as the sum of the variances of its components. Hence,

\[
\text{Variance } (r) = \text{variance } [E(r) + \beta r_M + e] = \text{Variance } (\beta r_M) + \text{Variance } (e)
\]

Notice, the variance of the expected rate of return has been dropped since it is assumed to be fixed – or, constant – prior to the investment. We may write the variance equation with the typical notation as the following

\[
\sigma^2 = \beta^2 \sigma_M^2 + \sigma_e^2 = \text{systematic risk + nonsystematic risk}
\]

noting that the beta term now relates the variance in the market to the variance of the asset. The variance of the asset determined by the market variance is termed systematic risk. The variance of the asset determined by the specific factors of the asset is termed nonsystematic risk.

The variance of the asset is composed of systematic risk and nonsystematic risk. The terms are sometimes referred to as non-diversifiable risk and diversifiable risk. In order to see the implications, consider what happens to the variance of a portfolio as the number of assets contained in the portfolio increases. As illustrated in Figure 5.1, the portfolio risk dramatically decreases as the number of assets increase. Empirical estimates for the stock market indicate that it does not take many stocks before the risk on the portfolio declines to the market (or, systematic) risk – 40 randomly selected stocks would typically be enough to do this, though often it can be much less.
The important point to observe is that the specific risk to an asset can be diversified away. Thus, those factors such as a CEO resigning influencing the variance of the actual rate of return can be eliminated with diversification. What cannot be eliminated is the risk of being in the market. When we say therefore that an investor is rewarded with a higher expected rate of return for taking on more risk, we actually mean a narrow definition of risk. The investor is not rewarded for taking on risk that could have been diversified away. This is like our example of a business owner justifying the higher rates of return because no one in the industry buys fire insurance. In that case, we did not believe that the higher return was in some sense 'justified'. The same is true here. An investor will not be rewarded with higher returns because they are too lazy to diversify their portfolio – thereby eliminating some of the risk. The investor will be rewarded, however, for purchasing an asset which varies more than the market itself.

5.4 Derivation of CAPM

The capital asset pricing model (CAPM) is derived by putting together all of the above pieces. First, when we assumed that all investors have the same information and form the same expectations, and use the mean-variance approach, they will hold the same optimum portfolio. This optimum portfolio will constitute the market portfolio. The assumptions do not include that the investors all have the same preferences. The separation theorem insured that we could separate the problem of identifying the optimum portfolio from investor preferences. Second, the relevant risk of an asset was determined to be only that part of its standard deviation influenced by the market risk. This was the risk that could not be diversified away. Now recall from Section 5.1 and Chapter 3, equilibrium will be established when the (relevant) risk adjusted rate of return for all assets are equal.

The assumptions of the CAPM are fairly strict and unrealistic to be sure. However, the resulting equilibrium condition has an intuitive appeal. The equilibrium condition can be stated formally in the following manner.

\[
\frac{E(r_1) - r_f}{\beta_1 \sigma_M} = \frac{E(r_2) - r_f}{\beta_2 \sigma_M} = \cdots = \frac{E(r_M) - r_f}{\beta_M \sigma_M}
\]

Actually, the condition can be simplified a bit by noticing that: (1) the beta for the market will be one – thus, the market moves one-to-one with the market, (2) the standard deviation of the market cancels out. This leaves us with the following condition.

\[
\frac{E(r_1) - r_f}{\beta_1} = \frac{E(r_2) - r_f}{\beta_2} = \cdots = \frac{E(r_M) - r_f}{\beta_M}
\]

Although it may not appear so, equation (5.5) is the equilibrium condition for asset prices. Consider the example of market equilibrium from section 5.1 and Chapter 3. The short-run market equilibrium was stated as quantity demanded equal to quantity supplied. The long-run equilibrium condition occurred when the risk adjusted rates of return were
equal. In similar fashion, equilibrium in asset markets is stated in terms of quantity demand and supply, but now we have a sort of long-run equilibrium condition in terms of the equality of risk adjusted rates of return. The relevant risk, again, being that part of risk that cannot be diversified away (i.e., the systematic or market risk).

Equation (5.5) is an equilibrium condition for asset prices, not just for returns. In order to see this, recall the definition for the rate of return (where \(v\) is the sum of the expected selling price and any dividends – the expected payoff included the sale price and any dividends).

\[
r = \frac{v - p}{p} \rightarrow p = \frac{v}{1 + r}
\]

From here, we can see that given the expected payoff \(v\), the rate of return calculated from equation (5.5) will determine the equilibrium price of the asset. If condition (5.5) is not satisfied, then asset prices will change. For example, assume that the risk adjusted rate of return on asset 1 exceeds that of asset 2.

\[
\frac{E(r_1) - r_f}{\beta_1} > \frac{E(r_2) - r_f}{\beta_2}
\]

Notice that this is similar to the case of the consumer in which the marginal utility per dollar of one good exceeds the other. In that case, we assume that the consumer will want to purchase more of the good with a higher marginal utility per dollar and less of the other until the marginal utilities adjust. Just as in the case of the consumer’s problem, the above inequality can be rewritten to indicate exactly what must adjust.

\[
\frac{E(r_1) - r_f}{E(r_2) - r_f} > \frac{\beta_1}{\beta_2}
\]

Here, since the betas are given, it is the left hand side that must adjust – or, more specifically, the expected rates of return. Given this inequality, the investor will increase (decrease) his demand for asset 1 (asset 2) causing the price to increase (decrease) until the equality is established. Notice the increase (decrease) in the price of asset 1 (asset 2) will cause the expected rate of return to fall (rise). This is the intuition behind the adjustment process to equilibrium and brings out the fact that equation (5.5) is indeed an equilibrium condition for asset prices.

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3 To be clear, the betas are equal to the covariance of the asset and market returns divided by the variance of the market,

\[
\beta_i = \frac{\sigma_{1M}}{\sigma_M^2}
\]
The equilibrium condition should be viewed as a general equilibrium. By this term we mean that all markets are being taken into account. In contrast, the market analysis conducted in an introductory microeconomics course - and, much in intermediate micro – is typically a partial equilibrium analysis since the ceteris paribus condition is normally invoked. In the context of partial equilibrium, one market is studied while others are held constant so that what happens in the one market (e.g., an increase in price) does not cause changes in other markets (e.g., increase in demand), which feedback to the initial market (e.g., increase in demand). The general equilibrium analysis allows for changes in one market to impact other markets, while those effects occurring in other markets have consequences as well. We can picture this in our asset markets as every demand schedule being drawn for given prices of other assets. Hence, a change in the price of one asset – causing a change in the expected rate of return – will lead to a shift in the demand schedule of other assets. The interrelationships between assets can be quite complex with many places for those to go wrong. The analysis here only provides an intuitive explanation of the equilibration process (i.e., the process of getting to the equilibrium) and bypasses the intricate technical issues involved. Instead, the focus is on studying the implications of the equilibrium condition.

In equilibrium, the expected rate of return for a particular asset will be completely determined by the risk-free rate of return, expected market rate of return, and the beta coefficient (i.e., systematic risk measure). The equilibrium condition (5.5) can be rewritten for an individual asset in the following form.

\[ E(r_f) = r_f + \beta_f[E(r_M) - r_f] \]

Hence, the expected rate of return on any asset will exceed the risk-free rate of return by the product of the systematic risk measure (i.e., beta) and the risk premium of the market as a whole. We are no longer concerned with the total risk of an individual asset – as measured by its standard deviation. Thus, we are not concerned with the total amount of variation that occurs in the expected return of the asset. Rather, we are concerned with how the rate of return of the asset relates to the market as a whole. This result derives from the previous section in which we demonstrated that some of the risk of an asset could be eliminated by diversification.

Equation (5.6) can be graphically depicted in a couple of different ways. Figure 5.2 demonstrates a typical graphical method – the line being termed the Security Market Line.

[insert Figure]

Graphically, we can read off the expected rate of return for any asset (or, portfolio) on the vertical axis for various levels of beta. The expected market rate of return is seen for a beta of 1 (i.e., the market moves one-to-one with the market). Assets lying above the security market line, such as Asset 1, are said to be underpriced. Assets lying below the line, such as Asset 2, are said to be overpriced. The implication being that Asset 1’s price will tend to rise, causing the expected rate of return to fall, moving us vertically
downward to the line. For Asset 2, the price will fall, increasing the expected rate of return, moving us vertically upward to the line.

The equilibrium price of an asset can now be stated more precisely. Recall our previous statement of price in terms of expected payoff and rate of return. We can now replace the rate of return \(r\) with the expected rate of return based on CAPM from equation (5.6).

\[
p = \frac{v}{1 + r_f + \beta[E(r_m) - r_f]} = \frac{v}{1 + r_f + \beta \theta}
\]

\((5.7)\) where \(\theta = E(r_m) - r_f\)

Hence, the equilibrium price is based on a present value calculation where the relevant discount factor is determined by the risk-free rate of return, beta, and the market risk premium. This will prove useful for us later. For now, it makes clear that CAPM is a theory of equilibrium prices – not just rates of return.

5.5. Examples of CAPM and Applications

5.5.1 Rates of Return and Prices

Example 1. Suppose the risk-free rate of return is 4% and the expected market rate of return 10%. Stock ABC has a beta of 2. Stock XYZ has a beta of .5.

a. Based on CAPM, what are the expected rates of return for each stock?

b. Assuming the stocks are priced according to CAPM, if you believed that the expected rate of return on ABC was 14%, then would you buy or sell short?

c. Assuming the stocks are priced according to CAPM, if you believed that the expected rate of return on XYZ was 8%, then would you buy or sell short?

Answer 1.

\[
E(r_{ABC}) = 4 + 2(10 - 4) = 16\%
\]

\[
E(r_{XYZ}) = 4 + .5(10 - 4) = 7\%
\]

b. In this case, the price of stock ABC – given the market’s determination of the expected payoffs – is such that it will give a rate of return of 16%. You believe – contrary to the market – that the return will be lower. Hence, you must disagree with the market on either the expected payoff (this might be the eventual sale price of the stock or on the relevant beta to be used). For example, you might believe that corporation ABC will not be paying the dividend that market believes it will pay – in fact, you think it will be lower. Hence, you will wish to sell this stock short – believing the price will fall in the future, once the market realizes its mistake.
c. Similar to the previous question, however this time you believe the market expects a lower payoff than you. Hence, you believe the stock price will rise leading you to buy now.

Example 2. Stock 1 has an expected return of 12% and beta of 1.0. Stock 2 has an expected return of 13% with beta of 1.5. The market’s expected return is 11% and the risk-free rate is 5%. According to the CAPM, which stock is a better buy? Plot the Security Market Line (SML) and the two stocks.

Answer 2.

\[ E(r_1) = 5 + 1(11 - 5) = 11\% \]
\[ E(r_2) = 5 + 1.5(11 - 5) = 14\% \]

Stock 1’s expected rate of return would lie above the security market line. Notice, in this case – unlike the previous problem – we are thinking in terms of whether the market has priced the stock according to CAPM. For stock 1, we would say that the current price is too low – or, the stock is underpriced according to CAPM. Thus, we would expect an increase in the demand for this stock, driving the price upward and pushing the expected rate of return down to 11%. Graphically, this would be shown by a downward, vertical movement of the expected rate of return toward the SML.

Stock 2’s expected rate of return is currently below the rate determined by CAPM. Hence, the stock is overpriced. We would expected a decrease in demand for stock 2 – hence, sales of stock 2 – driving the price down and the rate of return upward.

5.5.2 Capital Investments

Example 3. Suppose Silverado Springs Inc. is considering a new spring-water bottling plant. The business plan forecasts an internal rate of return of 14% on the investment. Research shows the beta of similar products is 1.3. Assume the risk-free rate is 4% and expected market return 12%. Based on CAPM, calculate the ‘hurdle rate’ (or, risk-adjusted discount rate) for this project. Should the project be done?

Answer 3.

The internal rate of return (IRR) is the discount rate that makes the present value of all future income just equal to the current price. Recall, the present value calculation is stated as

\[ PV = \frac{C_1}{(1 + i)} + \frac{C_2}{(1 + i)^2} + \frac{C_3}{(1 + i)^3} + \frac{C_4}{(1 + i)^4} + \frac{C_5}{(1 + i)^5} + \cdots + \frac{C_N}{(1 + i)^N} \]
Where $PV$ stands for present value, C’s stand for any payments received at various times (e.g., 1=1 year from now, 2=2 years from now, etc.), $i$ the interest rate (or, discount rate), and $N$ the last payment.

If we know the price ($P$) that this investment is currently selling for, then the $i$ become the internal rate of return that satisfies the following.

$$P = \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + \frac{C_3}{(1+i)^3} + \frac{C_4}{(1+i)^4} + \frac{C_5}{(1+i)^5} + \ldots + \frac{C_N}{(1+i)^N}$$

When dealing with bonds, the $i$ is called the yield to maturity.

In this problem, the firm knows the price of the project and estimates the future payments (or, profits) from investing in the project and how long those will last, thereby calculating the internal rate of return of 14%. Often this internal rate of return would be compared to the cost of borrowing (or, interest rate on the loan) to determine whether or not to do the project. However, notice, the internal rate of return does not take into account risk at all – more to the point, it does not take into account the “relevant” risk (i.e., systematic risk). Hence, after the development of CAPM and its widespread use, firms began to compare the internal rate of return to the expected rate of return as determined by CAPM.

$$E(r) = 4 + 1.3(12 - 4) = 14.4\%$$

Thus, given the risk of this project (as measured by its beta – the ‘relevant’ risk) the project should provide a rate of return of 14.4%. The firm should not do this project. In fact, noone should. If this occurs, then the price of the project will drop, and then – look at the equation for the IRR – the internal rate of return will rise (assuming the future payments, C’s, remain the same).

Example 4. An investor is considering a one year investment. The expected payoff of the investment is $1,000. This type of investment typically has a beta of 2. The risk-free rate is 5% and expected market return 15%. According to CAPM, what should the investor be willing to pay for this investment?

Answer 4.

In this case, we will use the present value formula to determine the price we’d be willing to pay for the investment. However, we want to use the ‘proper’ discount rate. The proper discount rate should take into account the relevant risk of the investment. Hence, if the investment was very risky, then the discount rate should be high, driving down the price of the investment (look at the present value formula). The relevant discount rate can be determined by CAPM.

$$E(r) = 5 + 2(15 - 5) = 25\%$$

Now, we simply plug this into the present value formula - we have made things simple for ourselves by assuming the investment lasts for just one year.
\[ P = \frac{\$1,000}{1 + .25} = \$800 \]

Notice, this equation follows directly from equation (5.7).

5.5.3 Regulation

Example 5. Suppose equityholders’ investment in a publicly regulated utility is $100 million and the beta is .5. If the risk-free rate is 5% and expected market rate is 11%, then what would be a ‘fair’ (according to CAPM) annual profit for the utility?

Answer 5.

\[ E(r) = 5 + .5(11 - 5) = 8\% \]

Notice, a utility company is normally considered to be a less risky investment. This fact explains why the beta is low. Recall, a beta of .5 implies that this asset will fluctuate half as much as the market – though, it is not risk free, that would be a beta of 0. Now, we have a ‘fair’ rate of return to target – fair being defined by CAPM standards. Based on an investment of $100 million, the investors should earn a total of $8 million in profits (i.e., $100 million x .08 = $8 million). The regulatory agency will use that number as a target and allow the company to set a price to meet that target.

Example 6. A consumer group is suing a cable company for price gauging. The company generated profits of $15 million last year with total stockholder equity of $100 million. The consumer group has looked at similar companies and determined that a relevant beta would be .8. Assuming the risk-free rate was 6% and expected market return 16%, does the consumer group have a case?

\[ E(r) = 6 + .8(16 - 6) = 14\% \]

Given its relevant risk, a ‘fair’ rate of return would have been 14%. Since the company made a 15% rate of return, there is a possibility of price gauging. If a judge wanted to set a penalty, then $1 million would seem appropriate (what the owners of the company made in excess of the ‘fair’ amount).

5.5.4 Performance Evaluation

Example 7. There exists a variety of methods for evaluating the performance of investment professionals. Suppose an investment professional constructed the following portfolio – market information has also been added.

<table>
<thead>
<tr>
<th></th>
<th>Portfolio</th>
<th>Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Return</td>
<td>16%</td>
<td>14%</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>20%</td>
<td>24%</td>
</tr>
</tbody>
</table>
Assume the average risk-free rate of return had been 6%.

a. Calculate the Sharpe ratio (or, return-to-variability ratio) for the portfolio and for the market. Based on this measure, how did the investment professional perform?

\[ RV = \frac{\bar{r}_p - \bar{r}_f}{\sigma_p} = \frac{16 - 6}{20} = .5 \]

For the market: \((14-6)/24 = .33\)

The professional did well.

b. Calculate the Treynor measure for the portfolio and the market. Based on this measure, how did the investment professional perform?

Like Sharpe, TM gives average excess return per unit of risk incurred, but it uses systematic risk instead of total risk.

\[ TM = \frac{\bar{r}_p - \bar{r}_f}{\beta_p} = \frac{16 - 6}{.80} = 12.5 \]

For the market: \((14-6)/1 = 8\)

The professional did well.

c. Calculate the Jensen measure for the portfolio. Based on this measure, how did the investment professional perform?

The Jensen measure is the average return on the portfolio over and above that predicted by the CAPM.

\[ JM = \bar{r}_p - [\bar{r}_f + \beta_p(\bar{r}_M - \bar{r}_f)] = 16 - [6 + .8(14 - 6)] = 3.6\% \]

The professional did well.
Example 8. Do the same as in example 7 for the following data. Assume the risk-free rate is still 6%.

<table>
<thead>
<tr>
<th></th>
<th>Portfolio</th>
<th>Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Return</td>
<td>35%</td>
<td>28%</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>42%</td>
<td>30%</td>
</tr>
<tr>
<td>Beta</td>
<td>1.2</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Answer 8.

I will let you answer this – very similar to the previous question. However, notice that previously all the performance measures came to the same conclusion – “professional did well”. See if that happens here. If they do not all agree, see if you can explain why.

5.6 Factor Models

Factor models attempt to explain rates of return - on individual assets or portfolios – on the basis of specified factors. In actual fact, there is not much theory behind factor models. A factor can be anything the analysis believes will explain the rates of return. For instance, an analyst might believe that Gross Domestic Product (GDP) provides a good explanation of what happens to rates of return. Thus, GDP would be a factor in the model. Alternatively, inflation (expected or unanticipated) could be a factor that explains rates of return and thereby asset prices. The analyst may choose to use a multifactor model in which both GDP and inflation are explanatory variables of rates of return.

The CAPM is a special type of factor model. The factor was the expected market rate of return – actually, the expected excess rate of return on the market (i.e., expected market return minus the risk-free rate). Since CAPM utilizes only one factor it would be considered a single factor market. In order to apply CAPM, since one cannot observe the rate of return on the entire market, a proxy is generally used for the expected market return. One common proxy is the S&P 500. Like the often quoted Dow Jones, the S&P 500 is an index of various stocks intended to capture the movement in the stock market as a whole. Thus, in order to apply CAPM, we would attempt to estimate the following:

\[
R = r_f + \beta (R_{S&P} - r_f) + e
\]

Better still, we could define each rate of return as the excess rate of return by taking the lone risk-free rate on the right hand side over to the left hand side.

\[
R = \beta R_{S&P} + e
\]

Estimating this equation would not be terribly difficult. We could gather historical data on the rate of return on our particular asset, the risk-free rate, and the rate of return on the

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4 Make sure that you understand what has been done in order to arrive at this equation. We will be using the notation of capital R frequently for the remainder of this chapter.
S&P 500. With the data in hand, we simply run a regression to estimate the beta. Of course, in order to make predictions, we would have to assume that the future will look similar to the past. The CAPM is special due to the severe assumptions that were put in place in order to arrive at it. In addition, notice that in estimating (5.8) we would want to test to make sure that there was no constant term – graphically, the line goes through the origin.

In general, factor models explain rates of return as linear functions of the factors chosen. In general, an N-factor model takes the following form

\[ R = \alpha + \beta_1 F_1 + \beta_2 F_2 + \cdots + \beta_N F_N + \epsilon \]

where \(F\)'s are the factors that determine the rate of return – and the \(R\) represents the asset’s excess return over the risk-free return. Thus, one collects data on the excess returns of the asset and the factors, then estimates the above equation. It should be clear that the theoretical foundations of factor models, by themselves, are weak. On the other hand, in applied work, factor models do not restrict the analysis by imposing a list of assumptions. These types of models represent statistical models, rather than theoretical models.

5.7 Arbitrage Pricing Theory

The CAPM can be viewed as one possible way to turn a factor model into a theory of asset prices. We have seen that the CAPM imposes strict – and unrealistic – assumptions in order to develop the theory. In 1976, Stephen Ross put forth an alternative - and in many ways less restrictive – theory of asset prices. The theory has come to be termed the Arbitrage Pricing Theory (APT) because it is derived primarily from the imposition of no arbitrage opportunities should exist in equilibrium. In the following, we will derive an APT for a portfolio. In this regard, it will be less general than CAPM. It is possible to demonstrate that under certain conditions the APT can determine at least approximately asset prices – hence, returns – for individual assets. By focusing on portfolios, we will be able to bring out the underlying logic of APT.

Arbitrage opportunities are narrowly defined in the finance literature. An arbitrage opportunity exists when an investor can achieve economic profits without risk and with a zero-investment portfolio. Economic profit means a return in excess of the risk-free rate. The zero-investment portfolio implies that the investor does not use any of their own wealth. In well-functioning markets we would suppose that these opportunities would be eliminated nearly immediately. The opportunities would result from a mispricing of assets. Here, the mispricing is not associated with the CAPM’s pricing rule.

The absence of arbitrage opportunities can be thought of as an application of the law of one price. Consider a situation in which you observe the same new book being sold at two different prices on two different internet auction sites. You could offer to sell the book on the high price site and purchase the book on the low price site. The money
you receive from selling will allow you to make good on the promise to purchase from
the other site – the remaining amount being your profit made without risk and without an
initial investment. An opportunity like this would surely not remain for long. In fact,
your actions alone could drive the price up on the one site and down on the other – since
without risk or need of initial investment, you would continue to buy low and sell high
until the price of the book was the same on both sites. Another example of an arbitrage
opportunity would be if one bank was offering a lower interest rate on loans than another
bank was offering on deposits. In this case, riskless profits could be earned by borrowing
at the first bank and depositing at the second – notice, no initial investment would be
required.

The underlying idea of the absence of arbitrage opportunities can be heard in the
old phrase ‘there’s no free lunch’ or ‘you don’t get something for nothing’. It is also
possible to view it as an application of the law of one price. In the book example, the
arbitrage opportunity was quite easy to spot, whereas other arbitrage opportunity might
require a computer program to uncover them. In only a slightly more complicated
case, we can illustrate this underlying idea with currencies. Suppose while reading the
financial pages you observe the following prices for currencies.

\[
\begin{align*}
&1 = 2€ \\
&1€ = 5₤ \\
&1₤ = .20
\end{align*}
\]

Is there an arbitrage opportunity? Suppose you take your $1 and buy 2€, then purchase
10₤, then purchase $2. You have doubled your money! By using the futures (e.g.,
buying and selling for future delivery) you could double your money without any initial
investment. This situation could surely not last for very long. In fact, the price of the
currencies will be forced to stand in a relationship to one another so that this opportunity
does not exist. You might imagine that if the list of currencies was quite long – just take
a look at the quoted exchange rates on the financial page – a quick glance will no longer
be sufficient to identify such an opportunity.

Like the CAPM, the APT introduces a condition that links prices together. For
the CAPM, the condition was that the excess return per unit of additional risk to the
portfolio would have to be equal – equation (5.5). The APT’s condition is that no
arbitrage opportunities exist. The condition that links prices together is very much like
the optimization condition in consumer theory. In the two good case, we have seen this
condition in the following form.

\[
\frac{MU_1}{P_1} = \frac{MU_2}{P_2} \Rightarrow \frac{MU_1}{P_1} = \frac{MU_2}{P_2}
\]

When two consumers are trading two goods, it must be the case that the marginal utility
per dollar of each good is equal. This condition implies that the prices of the two goods
stand in some strict relationship – that is, the relative prices must equal the relative
marginal utilities (or, to use the micro terminology, the marginal rate of substitution).
Continuing with the consumer problem, we notice that only relative prices have been established. For example, we may know that the price of good 1 must be double the price of good 2. However, we do not know the level of the prices – the price of good 1 could be $10 or $12 depending upon whether the price of good 2 is $5 or $6. Something else must be added in order to establish the level of the prices. Notice, the same problem arises in the CAPM – and will also arise in APT. How did CAPM determine the level of prices and not just the relative prices? I believe the answer is that it introduces a common factor – the market portfolio for CAPM. \{\{An interesting way to look at both CAPM and APT// need to consider this point further// Should the micro solution to the consumer problem be stated?\} The point, if developed further, might prove useful later when the quantity equation gets introduced.// ---&gt; Also, it might be a nice way of getting at the often neglected – except by Marx and Keynes – of the importance of the unit of account function of money. Take another look at Ch. 3 of Volume 1 in Marx and the first chapter of Keynes’s Treatise on Money. The point is that both placed an unusual emphasis on the role of the unit of account. &lt;&lt; Mundell appeared confused/astonished by Keynes’s emphasis on this in the Treatise&gt;&gt;, but thinking about it from this angle there should’ve been no surprise – it is absolutely necessary for Keynes to begin at this point, he needed a common factor.\ Yet, why was it important for Marx? The problem of not having a common factor arose even earlier than Ch. 3 for Marx in the discussion of accounting for profits. &lt;&lt; There, it was a matter of having some method of calculating the difference between heterogeneous inputs with a different output seen in the circuit C-C’; labor values, of course, would not do.&gt;&gt; Studying the problem from this angle should provide an answer.\}\}

Rates of return – hence, prices – can be derived from APT for either individual assets or portfolios. Deriving the rate of return for an individual asset, however, takes more work and gets a bit complicated. Even more, the APT derived rate of return for an individual asset is only an approximation – normally, a good theoretical approximation. In the following, we will derive the rate of return based on APT for a portfolio. The derivation for a portfolio tends to bring out the underlying ideas much more clearly than the case of an individual asset. We begin by noting that the APT is a factor model. Although the APT – unlike CAPM – is capable of incorporating many factors as explanatory variables for the portfolio rate of return, we will simplify once again by looking at a single factor. Furthermore, we will assume that the single factor is a market index.\^\(^5\) Hence, the statistical single factor model for an individual asset is stated as the following.

\[(5.10) \quad R_i = \alpha + \beta R_M + e \quad \text{where } R = r - r_f\]

The excess rate of return on the asset is linearly dependent upon a single factor. The random error term \(e\) can be interpreted as the non-systematic risk component as in section 5.3. With this interpretation, we know that for a well-diversified portfolio the

\[^5\) It is not necessary to assume the factor is the market portfolio. It could be an index of industrial production, interest rate on long-term government bonds, index of consumer confidence, etc. We make this particular assumption in order to compare to CAPM.\]
non-systematic risk will virtually disappear.\(^6\) Hence, the single factor model for the excess rate of return on a portfolio can be written as follows.

\[(5.11) \quad R_p = \alpha_p + \beta R_M \]

We should be clear, equation (5.11) is not the same as the CAPM equation. The constant alpha (\(\alpha\)) term prevents equation (5.11) from being read as equivalent to CAPM.

Does equation (5.11) imply the existence of any arbitrage opportunities? Recall that beta is a measure of the relevant risk of an asset or portfolio. In the CAPM, an asset with a beta of zero led to the conclusion that the rate of return on the asset would equal the risk-free return – after all, a beta of zero implies a risk-free asset in the sense that market movements do not affect the rate of return on the asset. Now, consider the case in which we constructed such a good portfolio that its beta was zero. Equation (5.11) implies that the excess return on the portfolio would equal alpha. Pulling out the risk-free rate and rearranging, we would observe that this riskless portfolio could have a rate of return exceeding the risk-free rate by the amount of alpha.

\[
\begin{align*}
R_p &= \alpha_p \\
&r_f - r_f = \alpha_p \\
r_f &= \alpha_p
\end{align*}
\]

However, this violates the assumption that the market will adjust until all arbitrage opportunities are eliminated. For example, an investor – without wealth – could borrow at the risk-free rate and invest in this special portfolio, earning a profit determined by alpha. If no arbitrage opportunities are allowed, then it must be the case that alpha is zero. The alpha term in equation (5.11) must be dropped.

\[(5.12) \quad R_p = \beta R_M \]

Somewhat ironically, we have arrived at the CAPM equation. To see this explicitly, we pull out the risk-free rate and take expected values.

\[
\begin{align*}
R_p &= \beta R_M \\
&r_f - r_f = \beta(r_M - r_f) \\
r_f &= \beta(r_M - r_f) \\
\Rightarrow
E(r_p) &= r_f + \beta[E(r_M) - r_f]
\end{align*}
\]

Using two different conditions, it is possible for the APT and CAPM to arrive at the same pricing equation.

It is easy to be misled by the derivation of APT offered here. The derivation has purposely been done to arrive at the common result. However, there are many points of contrast. First, the derivation of APT has been done for a portfolio rather than an individual asset – CAPM applies directly to both. As stated previously, it is possible to

\[\text{Footnote:} \quad \text{This is the real point of our simplification to the case of a portfolio. Bailey (2005) provides a nice – and, technically correct - derivation of the APT for an individual asset.}\]
derive a similar expression – equation (5.12) – for individual assets based on APT. In fact, one can argue that certainly alpha must be zero for most individual assets otherwise it would be impossible for a portfolio not to have a non-zero alpha. Second, at no point during the derivation of APT did we assume investors were alike or that they actually used a mean-variance analysis to construct the portfolio. The only behavioral assumptions APT requires is that investors form a well-diversified portfolio – ‘well’ being defined by the elimination of non-systematic risk – and that at least one person takes advantage of any arbitrage opportunities that happen to arise. Third, the CAPM is a single factor model, whereas APT is capable of incorporating more than one factor. Finally, one should be extremely careful when interpreting empirical tests of the two theories. In fact, many of the tests on CAPM should really be considered tests of APT.

5.8 Conclusion

{finish writing --- focus should be on the methodology, this will lead into the behavioral finance approach and begin to make the transition from micro to macro}