Chapter 4: Mean-Variance Analysis

Modern portfolio theory identifies two aspects of the investment problem. First, an investor will want to maximize the expected rate of return on the portfolio. Second, an investor will want to minimize the risk of the portfolio. The two aspects amount to the objective of maximizing the expected rate of return for any given, acceptable, level of risk. Alternatively the objective can be stated as: minimize the risk for any given, acceptable, level of expected return. For the purposes here, risk is associated with the variance - or more commonly, the standard deviation - of the portfolio.

The goal of section 4.1 is to construct the efficient frontier. Every point on the frontier will constitute a possible portfolio which meets the objective of maximum return for a given risk (or, minimum risk for a given return). Once the frontier is identified, then section 4.2 addresses the question of how an individual investor will choose among the various efficient portfolios. The investor's preferences describe how they are <u>willing</u> to trade-off higher returns for lower risks, while the efficient frontier describes how they are <u>able</u> to make the trade-off. Hence, without knowing the investor's preferences, we cannot determine which efficient portfolio will be chosen. Our job is only to construct the efficient frontier, then let the investor decide where they would like to be on it.[1] Section 4.3 introduces a risk-free asset, which turns out to have important implications. The introduction of a risk-free asset allows the investor to separate the question of identifying the optimum risky portfolio from his/her own preferences – a result known as the Separation Theorem. Section 4.4 recounts the context and original development of the separation theorem.

A word of caution is in order before beginning. In order to keep the calculations in the examples down to a manageable number, we will be constructing a portfolio from only two risky assets (and adding one risk-free asset in section 4.3). The results obtained will generalize to any number of assets. The results and their interpretation will be stated in the generalized form. In doing so, the explanations may appear to strain the actual numerical results.

4.1 Construction of the Efficient Frontier

Previously, we had been concerned with various ways in which to think about the determination of asset prices (e.g., treat assets as stocks or flows). Since asset prices are forward looking, the question of how expectations are formed kept arising.[2] A high asset price implied – holding other things such as risk of default and liquidity constant – a high expected payoff. The focus was on buying an asset with a high expected payoff and as others did this, then the price would be driven upward. Harry Markowtiz was the first to systematically elevate the issue of risk to a position on par with expected return. The efficient frontier illustrates the trade-off that exists between expected return and risk. In the construction of the frontier, we will discover the

[1] The connection to consumer theory of standard microeconomics should become abundantly clear. The consumer attempts to maximize utility as described by his/her indifference curves. The indifference curves illustrate how the consumer is <u>willing</u> to trade one good for another. The budget constraint describes how the consumer is <u>able</u> to trade one good for another – given relative prices of the goods.

[2] Asset prices are forward looking in the sense that they depend upon what the future is expected to be – i.e., what state of the world actually comes to pass.

benefit of diversification. A simple example will be used to demonstrate the procedure of constructing the efficient frontier.

The efficient frontier should be constructed from a large number of possible assets. However, in order to illustrate the basic procedure, it will be useful to limit the scope of the assets to two. Suppose an investor has a total of $100,000 to invest in a stock and/or bond – it might be preferable to think of these as index funds. The investor must decide how much of the $100,000 to invest in the stock and how much to invest in the bond. This decision will determine the investor's portfolio (i.e., collection of assets). The characteristics of each asset are given in table 5.1.

Table 5.1

	Stock	Bond
E(r)	7.50%	5.00%
σ	12.50%	5.00%
ρ	-1	

Suppose the investor held only the bond. The expected return would be 5% with a standard deviation of 5%. Dissatisfied with this return and willing to take on more risk if necessary, the investor sells some bonds and invests in the stock. Suppose the investor's new portfolio contained 30% stocks and 70% bonds. What is the expected rate of return on the investor's portfolio? Recall that the expected rate of return on a portfolio is merely the weighted average of the individual rates of returns – where the weights are the percentage of the asset in the portfolio.

(4.1) $E(r_p) = W_s E(r_s) + W_B E(r_B)$

The *W*'s represent the weights (or, percentage of the asset in the portfolio) and subscripts *S* and *B* refer to stock and bond respectively. In our example, the expected rate of return on the portfolio is calculated as follows.

(4.2) $E(r_p) = W_s E(r_s) + W_B E(r_B) = (.3)(7.5) + (.7)(5) = 5.75\%$

Hence, the investor has been able to increase the expected rate of return on the portfolio by holding some stock.

What was the cost of obtaining the higher expected rate of return? The investor may have believed that he would have to take on more risk (i.e., higher standard deviation) to obtain a higher expected return. However, did risk increase? In order to calculate the standard deviation of the portfolio we begin by calculating the variance of a portfolio.[3]

(4.3) $\sigma_P^2 = (W_s\sigma_s)^2 + (W_B\sigma_B)^2 + 2W_sW_B\sigma_s\sigma_B\rho$

[3] The derivation of this equation is given in the mathematical and statistical appendix.

Recall, the Greek letter rho (ρ) is the correlation coefficient – which when multiplied by the two standard deviations equals the covariance between the two assets. The important point to notice about the above equation for the variance is that unlike the expected rate of return it is not – at least not always – the simple weighted average of the individual variances.[4] In our example, the variance of the portfolio composed of 30% stock and 70% bonds is the following.

(4.4)
$$\sigma_P^2 = (W_s\sigma_s)^2 + (W_B\sigma_B)^2 + 2W_sW_B\sigma_s\sigma_B\rho$$
$$= [(.3)(12.5)]^2 + [(.7)(5)]^2 + 2(.3)(.7)(12.5)(5)(-1) = .0625$$

The standard deviation of the portfolio (i.e., our measure of risk) is the square root of the variance.

(4.5) $$\sigma_P = \sqrt{\sigma_P^2} = \sqrt{.0625} = .25\%$$

By adding an asset (i.e., the stock) with a higher rate of return and risk to his bond-only portfolio our investor has been able to increase the expected rate of return – not very surprising – and reduce the overall risk of the portfolio – this is very surprising. Here we see our first indication of the power of diversification. In addition, we see that the bond-only portfolio was not a very efficient portfolio. That is, there is at least one portfolio available – the one we used – with higher expected return and lower risk.

Practice 1. Suppose the investor liked what he saw happened so much that he decided to place $70,000 (or, 70%) in the stock and $30,000 (or, 30%) in the bond. What is the expected rate of return and standard deviation (risk) of this portfolio?

{{{Answer 1.

$$E(r_p) = W_sE(r_s) + W_BE(r_B) = (.7)(7.5) + (.3)(5) = 6.75\%$$

$$\sigma_P^2 = (W_s\sigma_s)^2 + (W_B\sigma_B)^2 + 2W_sW_B\sigma_s\sigma_B\rho$$
$$= [(.7)(12.5)]^2 + [(.3)(5)]^2 + 2(.7)(.3)(12.5)(5)(-1) = 52.56$$

$$\sigma_P = \sqrt{\sigma_P^2} = \sqrt{52.56} = 7.25\%\ \}\}\}$$

Suppose our investor having tried three different portfolios (bond-only, 30% stock and 70% bond, 70% stock and 30% bond) realized that the results were changing in uncertain ways. Moving from the bond-only to 30%-70% stock-bond portfolio the rate of return increased and the risk decreased. Moving more into stock, the rate of return continued to increase, but so did the risk. Table 5.2 might help the investor begin to see some pattern in the various portfolios.

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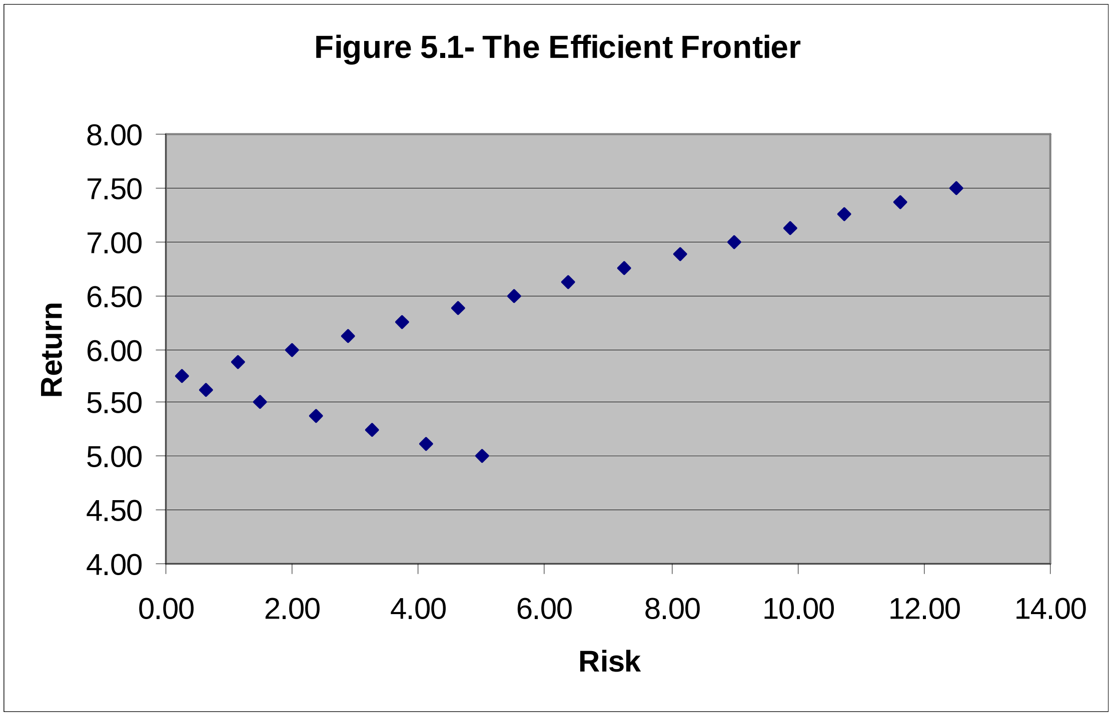
[4] The appendix contains further commentary on this point.

TABLE 5.2

Stock	Bond	Return (%)	Risk (%)
100%	0%	7.5	12.5
95%	5%	7.375	11.625
90%	10%	7.25	10.75
85%	15%	7.125	9.875
80%	20%	7	9
75%	25%	6.875	8.125
70%	30%	6.75	7.25
65%	35%	6.625	6.375
60%	40%	6.5	5.5
55%	45%	6.375	4.625
50%	50%	6.25	3.75
45%	55%	6.125	2.875
40%	60%	6	2
35%	65%	5.875	1.125
30%	70%	5.75	0.25
25%	75%	5.625	0.625
20%	80%	5.5	1.5
15%	85%	5.375	2.375
10%	90%	5.25	3.25
5%	95%	5.125	4.125
0%	100%	5	5

What are the implications of this analysis? First, think of beginning with the bond-only portfolio. The expected rate of return is 5% with a standard deviation of 5%. Now, suppose the investor decides to allocate 95% of the total investment in the bond and 5% in the stock. The expected return will <u>increase</u> to 5.13% and the standard deviation (risk) will actually <u>decrease</u> to 4.13%! Surely, the bond-only portfolio is not efficient. In other words, if we can find another portfolio with a higher expected return and the same - or, smaller - standard deviation, then the original portfolio should not be chosen regardless of the investor's preferences. Second, notice that as we move up the table from the bottom row (where only the bond is held) the risk initially declines, then reaches a minimum, then begins to increase. All along this upward movement in the table, the expected rate of return is increasing.

The implication can be clearly seen in Figure 5.1. The efficient frontier consists of only the "upper" portion of the line. This upper portion consists of portfolio's in which the expected rate of return can be increased only at the cost of an increase in the standard deviation (risk). We can graphically pick the efficient portfolios, choose a given level of risk (i.e., pick a point on the horizontal axis), then move upward until the portfolio with the highest expected rate of return (i.e., move up from the horizontal axis to the highest point on the curved line) is reached. Alternatively, for a given expected rate of return (i.e., pick a point above the minimum risk, on the vertical axis), choose the portfolio which minimizes risk (i.e., move to the right until you hit the curved line).



Doing this for all possibilities, we would find that the efficient portfolios lie on the upper part of the schedule.[5] Though somewhat misleading, the efficient frontier may refer to the entire schedule. Understood to mean this, we find that in a two asset case, each asset will lie on the frontier. In general, when the number of possible assets exceeds two, no individual asset will be on the frontier – again, the power of diversification.

Practice 2. Suppose an investor constructs a portfolio with 20% in stock and 80% in bonds. The investor has calculated the expected return on the stock to be 10% with a standard deviation of 25%. The expected return on the bond is 6% with a standard deviation of 12%. In addition, the correlation between stock and bond returns is zero. Calculate the expected return and standard deviation (risk) of this portfolio.

{{{Answer 2. The expected rate of return on the portfolio will be:

$E(r_P) = (.2)(10) + (.8)(6) = 6.8\%$

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[5] The appendix provides the formula for the minimum standard deviation. Given a correlation of -1, the minimum is quite straightforward to calculate – it is zero. In addition, the appendix provides the equation for the weights to choose in order to minimize the standard deviation. For example, in the case of a correlation of -1, the minimum standard deviation (zero) will occur when the portfolio contains a percentage of bonds equal to

$\frac{\sigma_s}{\sigma_s + \sigma_B}$. In our example, this would be approximately 12.5/(12.5+5) = .71. Thus, in order to reduce the standard deviation to zero, the investor should hold a portfolio consisting of 29% stocks and 71% bonds.

The variance of the portfolio is:

$$\sigma_P^2 = (W_s\sigma_s)^2 + (W_B\sigma_B)^2 + 2W_sW_B\sigma_s\sigma_B\rho$$
$$= [(.2)(25)]^2 + [(.8)(12)]^2 + 2(.2)(.8)(25)(12)(0) = 117.07$$

note, since the correlation is zero, the last term becomes zero as well.

The standard deviation (risk) of the portfolio is the square root of the variance:

$\sigma_P = 10.82\%$

In this example, we have been able to reduce the risk below the risk of holding only the bond while increasing the rate of return on the portfolio above the bond rate of return.}}}

What is the impact of the correlation coefficient? We will use the example in *Practice 2* in order to study the role of the correlation coefficient.[6] However, we change the value of the correlation coefficient from -1 to +1. Notice that as long as the correlation coefficient is less than +1, diversification can lead to more efficient portfolios. In the unlikely event of the correlation coefficient being exactly +1, then the variance of the portfolio is equal to the weighted average of the individual variances – just like the expected rate of return of the portfolio. This implies that there will always be a trade-off between risk and return. The important point, however, is that the benefits of diversification do not hinge on a negative correlation between assets in a portfolio. However, clearly, there are greater benefits to be had with a negative correlation. The only condition for the benefits of diversification is that the assets do not move perfectly together (or, the case of the correlation coefficient being exactly +1).

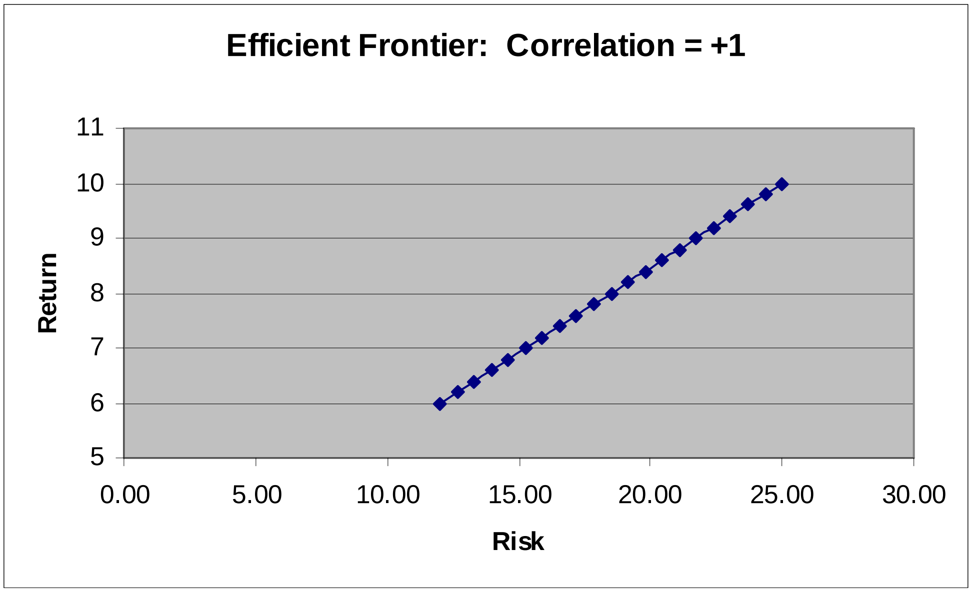
The table indicates that the closer the correlation coefficient comes to -1, the greater will be the benefits of diversification. This can be seen by choosing any portfolio containing both stocks and bonds (i.e., choose a row in the table). Now, move across the row from a correlation of +1 to -1. The expected rate of return of each portfolio in the row remains the same (this should be clear from the calculation for expected rate of return of a portfolio). However, the standard deviation (risk) continues to decrease! The far right hand side of the table assumes that stocks and bonds have a perfect negative correlation. In the case of perfect negative correlation (i.e., -1) between two risky assets, it is always possible to construct a portfolio with zero standard deviation (risk). In this particular case, if the portfolio contained approximately 32.4% stock and 67.6% bonds, then the standard deviation (risk) would be zero. The figures present a graphical representation of these ideas. Moving from the figure with a correlation coefficient of +1 –

[6] Recall that the correlation coefficient measures the relationship between two variables – in our case, the variables are the expected rates of return of the two assets. The correlation coefficient is closely related to the covariance between two variables. However, the correlation coefficient is easier to interpret. The closer a correlation coefficient is to -1, the stronger the inverse relationship between the two variables. In the case of exactly -1, we say that the two variables are perfectly inversely related meaning that they always move in opposite directions. The closer the correlation coefficient is to +1, the stronger is the positive relationship (i.e., they tend to move in the same direction). A correlation coefficient close to zero implies a lack of relationship between the two variables.

where the efficient frontier is a straight line indicating that the portfolio rate of return <u>and</u> standard deviation are simply weighted averages – to the figure based on a -1, the efficient frontier gets pulled toward the vertical axis – illustrating that risk is falling.

Table 5.3

Allocation		**Correlation = +1**		**Correlation = +0.5**		**Correlation = 0**		**Correlation = -0.5**		**Correlation = -1**	
Stocks	Bonds	Portfolio Return	Portfolio Risk	Portfolio Return	Portfolio Risk	Portfolio Return	Portfolio Risk	Portfolio Return	Portfolio Risk	Portfolio Return	Portfolio Risk
1	0	10	25.00	10	25.00	10	25.00	10	25.00	10	25.00
0.95	0.05	9.8	24.35	9.8	24.06	9.8	23.76	9.8	23.46	9.8	23.15
0.9	0.1	9.6	23.70	9.6	23.12	9.6	22.53	9.6	21.92	9.6	21.30
0.85	0.15	9.4	23.05	9.4	22.20	9.4	21.33	9.4	20.41	9.4	19.45
0.8	0.2	9.2	22.40	9.2	21.30	9.2	20.14	9.2	18.91	9.2	17.60
0.75	0.25	9	21.75	9	20.42	9	18.99	9	17.44	9	15.75
0.7	0.3	8.8	21.10	8.8	19.55	8.8	17.87	8.8	16.01	8.8	13.90
0.65	0.35	8.6	20.45	8.6	18.71	8.6	16.78	8.6	14.61	8.6	12.05
0.6	0.4	8.4	19.80	8.4	17.89	8.4	15.75	8.4	13.27	8.4	10.20
0.55	0.45	8.2	19.15	8.2	17.10	8.2	14.77	8.2	12.00	8.2	8.35
0.5	0.5	8	18.50	8	16.35	8	13.87	8	10.83	8	6.50
0.45	0.55	7.8	17.85	7.8	15.63	7.8	13.04	7.8	9.79	7.8	4.65
0.4	0.6	7.6	17.20	7.6	14.96	7.6	12.32	7.6	8.94	7.6	2.80
0.35	0.65	7.4	16.55	7.4	14.34	7.4	11.72	7.4	8.32	7.4	0.95
0.3	0.7	7.2	15.90	7.2	13.78	7.2	11.26	7.2	7.99	7.2	0.90
0.25	0.75	7	15.25	7	13.28	7	10.96	7	7.99	7	2.75
0.2	0.8	6.8	14.60	6.8	12.85	6.8	10.82	6.8	8.32	6.8	4.60
0.15	0.85	6.6	13.95	6.6	12.50	6.6	10.87	6.6	8.94	6.6	6.45
0.1	0.9	6.4	13.30	6.4	12.24	6.4	11.09	6.4	9.79	6.4	8.30
0.05	0.95	6.2	12.65	6.2	12.07	6.2	11.47	6.2	10.83	6.2	10.15
0	1	6	12.00	6	12.00	6	12.00	6	12.00	6	12.00


Efficient Frontier: Correlation = +1
11
10
9
8
7
6
5
Return
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10.00
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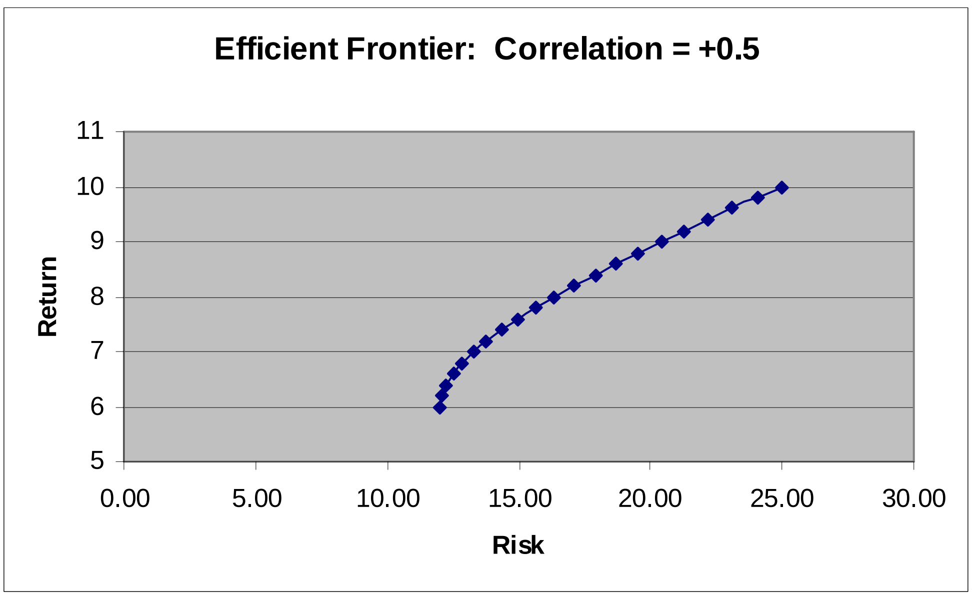
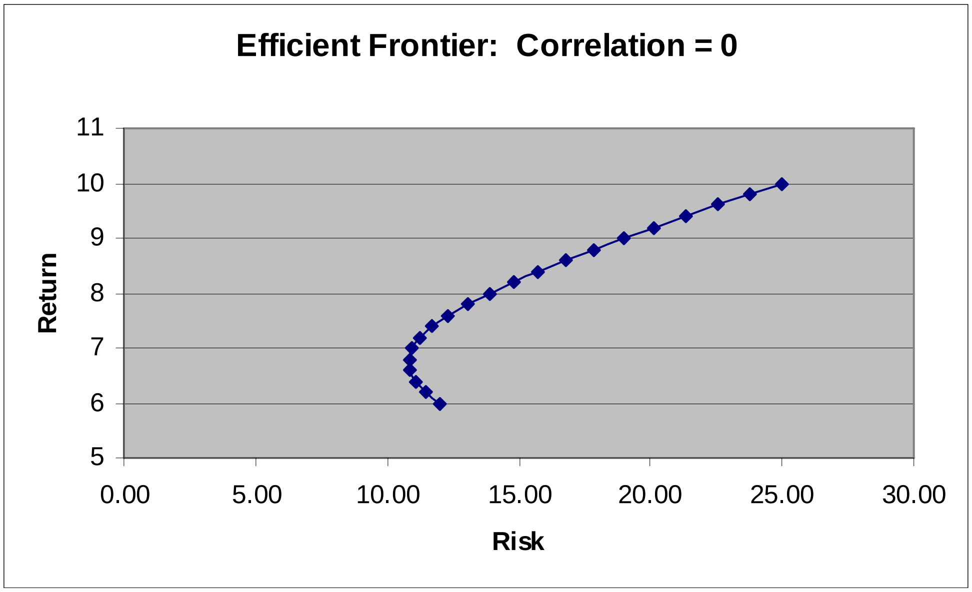
25.00
30.00
Risk


Efficient Frontier: Correlation = +0.5
11
10
9
8
7
6
5
Return
0.00
5.00
10.00
15.00
20.00
25.00
30.00
Risk


Efficient Frontier: Correlation = 0
Return
11
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9
8
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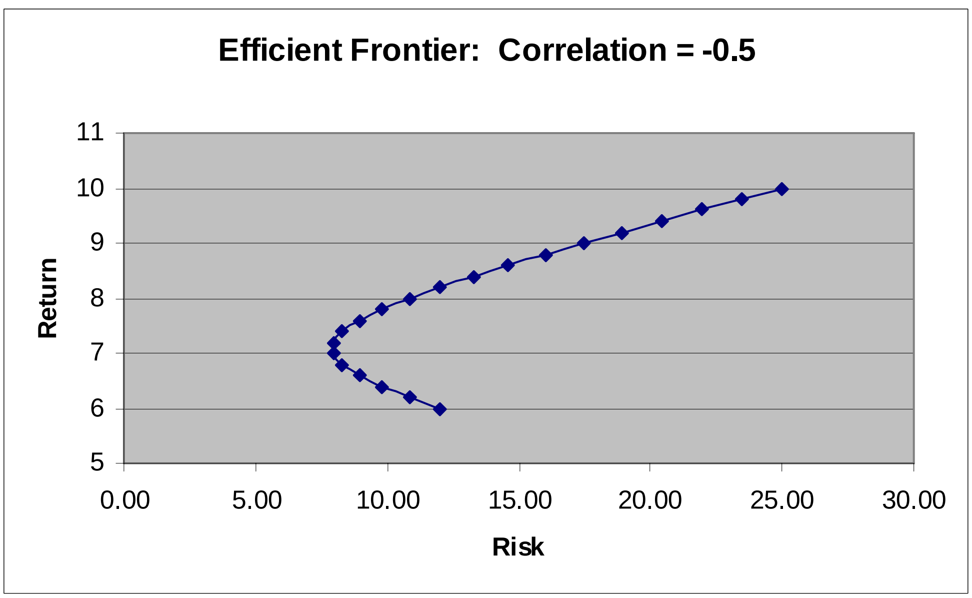
25.00
30.00
Risk


Efficient Frontier: Correlation = -0.5
Return
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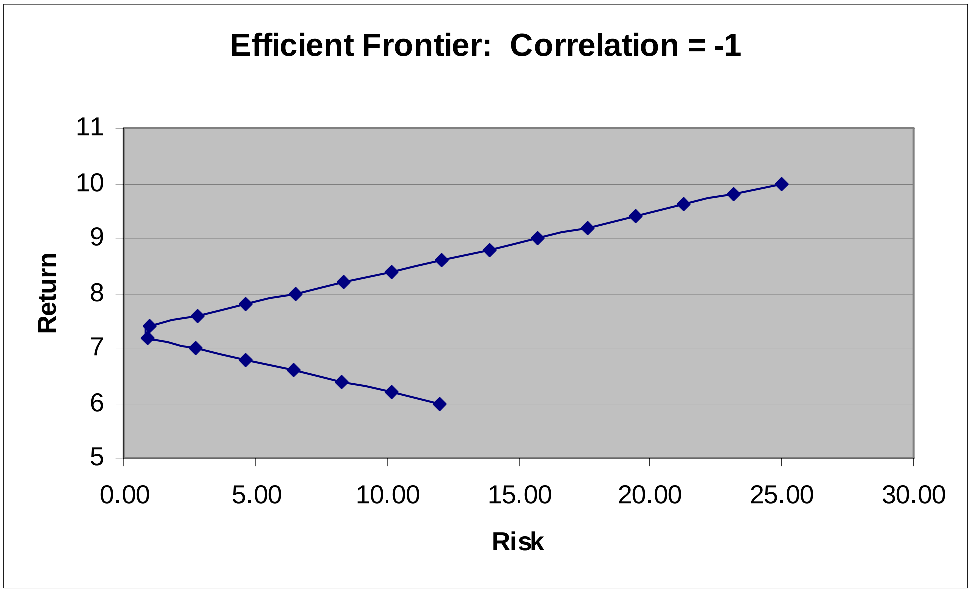
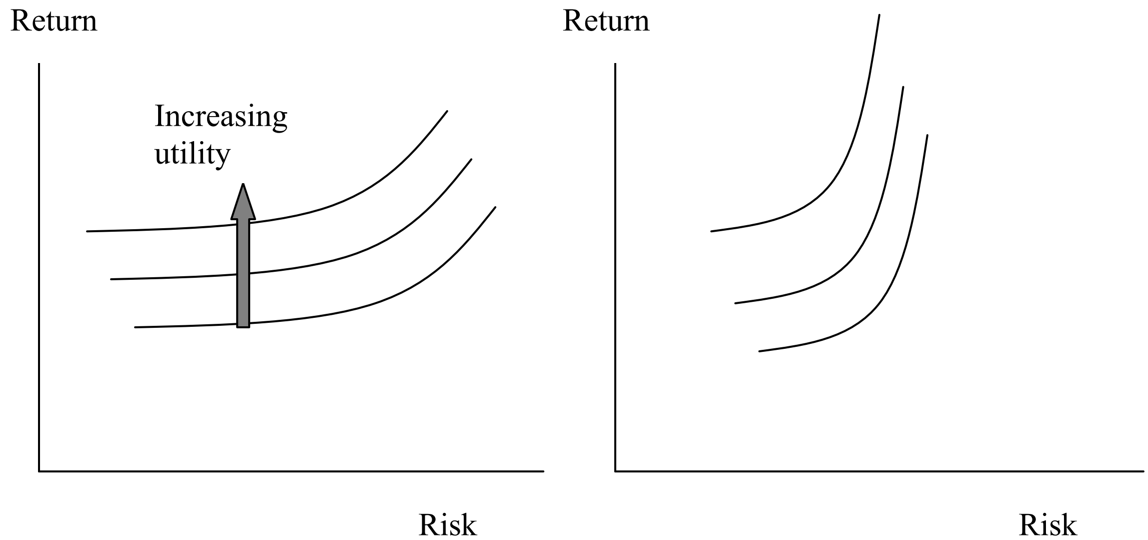


The efficient frontier allows us to find the portfolios with maximum return for given risk. The frontier acts much like a consumer's budget constraint in indicating how the investor is able to trade-off risk for return. Just as the case of the consumer's budget constraint, choosing to be inside the efficient frontier leads to an inefficient outcome. The question now is to begin to address the question of how the investor should choose between the efficient portfolios.

4.2 Illustrating Preferences for Risk and Return

Exactly which efficient portfolio should an investor choose? The answer will depend upon the investor's particular preferences. Is the investor willing to take on more risk in order to gain a higher expected rate of return? The investor's preferences can be illustrated with a set of indifference curves. Along any particular indifference curve, the investor has the same amount of utility (or, satisfaction). The indifference curves will slope upward. This indicates that in order to leave the investor with the same utility, the investor must be compensated with higher expected rates of return for greater levels of risk. A higher indifference curve is always better. This simply demonstrates that the investor will achieve a higher level of utility since his/her expected rate of return can be higher for any given level of risk. Alternatively, you may read the indifference curves horizontally as stating a lower risk for any given level of expected return.

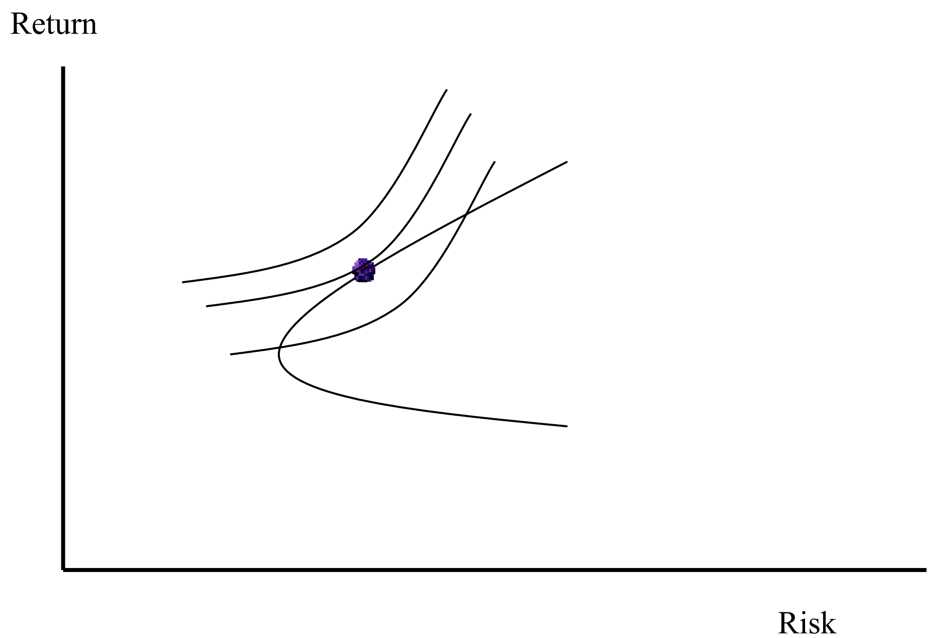


“Young Executive” “Little Old Lady”

The figures above present two ‘types’ of investor’s preferences. The indifference curves for the ‘young executive’ demonstrate that he requires little additional expected return for taking on more risk. The ‘little old lady’ on the other hand requires a large increase in her expected rate of return for taking on additional risk. We can think of reasons why these ‘types’ of investors view the risk-return trade-off in their particular way. The ‘young executive’ has a steady income in the form of a salary and a long investment time horizon, which allows him to view the cost of additional returns somewhat mildly. The ‘little old lady’ on the other hand may not have another source of income and has a short investment horizon (note, not necessarily a short time left to live, but rather needs to be cashing out of some of her investments soon). The important general point is that for both types of investors, they still hope to get on the highest indifference curve possible. As we move up indifference curves, the investor achieves higher expected rates of return for the same – or, less – risk. Thus, the higher indifference curves are superior regardless of preferences.[7]

We can now turn to the question of which efficient portfolio the investor *should* choose. The investor’s problem is to maximize the expected rate of return and minimize the risk of the portfolio subject to the available efficient portfolios. Graphically, the investor is attempting to reach the highest indifference curve possible, given the constraint of the efficient frontier. The optimum risky portfolio for an individual investor will be given by the point at which the indifference curve (illustrating the investor’s preferences for the risk/return trade-off) is just tangent to the efficiency frontier (illustrating all possible efficient portfolios) - this is point **O** on the graph.

[7] Throughout we assume investors are risk-averse. It is perfectly possible to treat investors as risk-neutral or even risk-loving, but the cost of the complications that arise from those assumption would seem to far outweight the benefits for us at this point.



Although the investor may like to be on the higher indifference curve, it is simply not possible given the characteristics of the risky assets. On the other side, any other point on the efficiency frontier results in a lower indifference curve, thus a lower level of utility. The optimum risky portfolio will be different for investors with different risk-return preferences – hence, different shapes of their indifference curves such as the 'young executive' and 'little old lady'. The next section demonstrates that this result may not hold in all cases.

4.3 The Separation Theorem

A risk-free asset can be introduced into our portfolio. The important implications of this introduction may appear surprising. The first task, however, will be to deal with the technical aspect. This can be done in a quick and dirty way. Reconsider the stock-bond characteristics of *Practice 2*. We can work with the case of a zero correlation coefficient along with a 50% composition of stocks and bonds (the rate of return is 8% and standard deviation 13.87%, see the table) - call this the 'risky' portfolio. Now, suppose you could buy a risk-free treasury bill paying 5%. A portfolio (call it the "complete portfolio") can be composed of the risk-free asset (treasury bill) and the risky assets (stocks and bonds). The decision concerns how much of your total amount (e.g., $100,000) is allocated towards each type of asset. The resulting complete portfolio will have an expected rate of return and standard deviation. We can illustrate the procedure with a little practice problem.

Problem 3. An investor has $100,000 to invest. The investor has chosen to construct a portfolio containing 25% of a risk-free treasury bill (5% rate of return and zero standard deviation) and 75% of risky assets. The risky portion of the complete portfolio is composed of 50% stock (10% expected rate of return and 25% standard deviation) and 50% bond (6% rate of return and 12%

standard deviation). Calculate the expected rate of return and standard deviation of the complete portfolio. Notice, we have already calculated the expected rate of return and standard deviation for the risky portfolio. All that this example requires is to use the previous formulas with the two types of assets: risk-free and risky portfolio.

{Answer 3.

$$E(r_C) = W_P E(r_P) + W_f E(r_f) = (.75)(8) + (.25)(5) = 7.25\%$$

where subscript C stands for the Complete portfolio (including the risk-free asset), P for the risky portfolio, and f for the risk-Free asset.

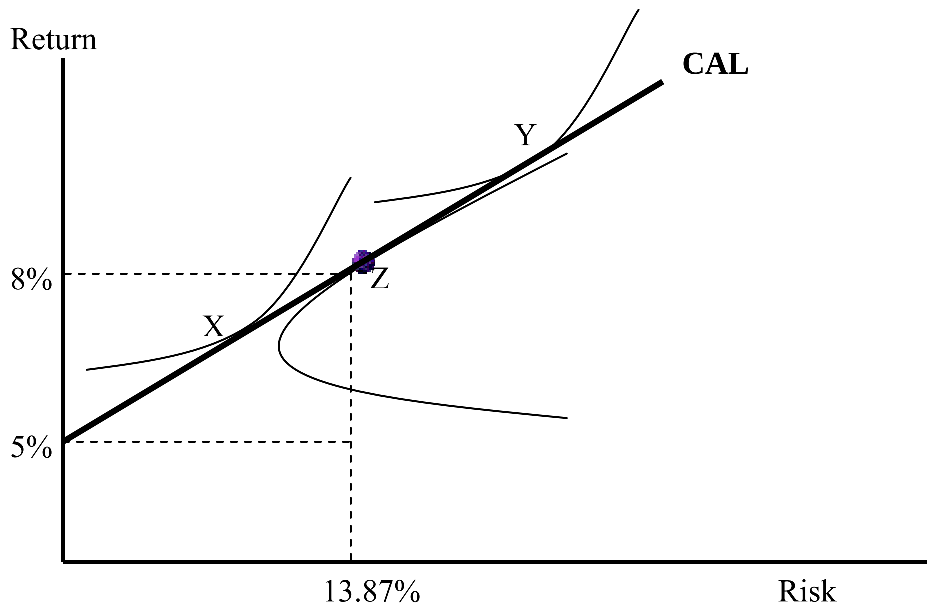
$$\sigma_C^2 = (W_P \sigma_P)^2 + (W_f \sigma_f)^2 + 2W_P W_f \sigma_P \sigma_f \rho$$
$$= [(.75)(13.87)]^2 + [(.25)(0)]^2 + 2(.75)(.25)(13.87)(0)(0) = 108.21$$

And,

$$\sigma_C = \sqrt{\sigma_C^2} = \sqrt{108.21} = 10.4\%$$

}}}

The intuition of the analysis can be seen with the aid of a graph. We will avoid some of the technical details. Essentially, we are forming a linear combination of a risk-free asset and a risky portfolio in order to construct a complete portfolio. If we began with only the risk-free asset, then - using our numerical example - the complete portfolio would have a rate of return of 5% and zero standard deviation (this is the point on the vertical axis). On the other hand, if the complete portfolio did not contain the risk-free asset, then the expected rate of return would be 8% with a standard deviation of 13.87% - this is point Z on the graph (note, this has been drawn as the optimum risky portfolio for convenience). By varying the percentage of our total investment allocated to the risk-free asset and risky portfolio, the expected rate of return and standard deviation will be given by a straight line between the point on the vertical axis representing the risk-free asset and point Z, the risky portfolio. The point X represents the complete portfolio of *Practice 3* where the expected rate of return turns out to be 7.25% with a standard deviation of 10.4%.



The capital allocation line (CAL) represents the complete portfolio for various allocations between the risk-free asset and risky portfolio. The slope (rise/run) of the CAL is given by the following:

$$\frac{E(r_P) - r_f}{\sigma_P} = \frac{8\% - 5\%}{13.87\%} = .217$$

The slope is sometimes called the "reward-to-variability ratio" (or, Sharpe ratio). This slope equals the increase in expected return than an investor can obtain per unit of additional standard deviation (risk). The portion of the line between the risk-free rate of return and point Z is where the investor is lending a portion of his/her total investment to the default free borrower (i.e., the government). This is illustrated by point X. What about the portion of the line beyond point Z. This part of the line would result if the investor could *borrow* at the risk-free rate, then purchase more than 100% of his/her own investment money into the risky portfolio. This is illustrated by point Y. Now of course, except for the government, an investor cannot actually borrow at this risk-free rate. If we wanted more realism, then the CAL will have a kink at point Z indicating the slope of the line gets flatter as the interest rate on a loan is greater than the risk-free rate. You can also think of the portion of the CAL beyond Z as indicating that the investor is buying on margin – still though, the cost of doing so will exceed the risk-free rate.

What would have happened if a risky portfolio with a higher expected rate of return and standard deviation had been chosen? For example, suppose you had chosen a risky portfolio composed of 75% stock and 25% bond. Using our previous numbers, the expected rate of return would be 9% with a standard deviation of 18.99% (this can be seen in the previous table). The slope of the CAL would decline slightly. This indicates a lower expected rate of return for an additional unit of risk. You could continue along this path - choosing various risky portfolios and drawing the CAL. Which CAL would be best? The one with the highest slope! This will occur when the CAL is just tangent to the efficient frontier. This is the one we have drawn in the figure.

We have everything needed to state and apply the separation theorem. Before doing so, consider the path taken to get to this point. Imagine that you are an investment counselor. It is your job to set up a financial portfolio for a client. Your first step would be to calculate the expected rate of return and standard deviation for every possible risky financial asset. The probabilities associated with the expected value and variance formulas can be determined by (a) looking at the past price data for each asset, (b) considering the financial position (e.g., use of some accounting and financial ratios) and prospects (e.g., a new CEO, a new product line, competition, etc.), and/or (c) subjective measures (e.g., gut feeling). This is, of course, a fairly daunting task.

The second step is to form various combinations of the risky assets in order to define the efficient frontier (a fairly easy process with a good computer). Once this is done, you could stop here with the calculations and attempt to understand your client's personal preferences between expected rate of return (a good thing) and risk (a bad thing). This is where you would be attempting to discover your client's particular indifference curves. Having done so, you can advise the client to buy a particular portfolio of financial assets. However, and this is where the separation theorem comes in, suppose that you have identified what you consider to be an excellent portfolio – an 'optimum' portfolio of risky assets. Would you really want to advise your client to choose another portfolio simply because of their personal preferences for return/risk? Shouldn't there be a way to purchase the excellent portfolio and still meet your client's personal preferences? This takes us to the next step.

The third step is to find a risk-free asset. The U.S. Treasury Bill serves this role nicely. However, you could choose something like a money market mutual account for your client where the rate of return was slightly higher while the risk remains pretty near zero. Whichever risk-free asset you choose, you must now construct the Capital Allocation Line (CAL). You do this by combining - in various amounts - the risky portfolios on each point of the efficient frontier with the risk-free asset. Identify the CAL with the highest slope. This is the one that gives the greatest expected return for each additional unit of risk. It is also the one that is just tangent to the efficient frontier.

The point - labeled Z - on the efficient frontier that is just tangent to the CAL <u>is</u> that <u>excellent</u> (or, optimal risky) portfolio. This is the risky portfolio that you should advise <u>all</u> your clients to hold regardless of their personal preferences for return/risk. It doesn't matter if your client is the little old lady or the young executive. The only difference between the clients, reflected in their preferences, will be how much of their total investment to allocate to the risk-free asset and how much to this optimal risky portfolio. For the little ol' lady, you may advise her to have a complete portfolio like point X in the figure. In this case, she would be holding some portion of her wealth in risk-free government T-bills. For the young executive, you might encourage the young investor - willing to take on even more risk in the hopes of higher returns - to borrower at the risk-free interest rate in order to purchase more of the optimal risky portfolio than what he/she could buy with their current wealth. Graphically, you are moving the investor up and to the right along the CAL to a point like Y. But notice, you are still advising to buy into only the identified optimal risky portfolio. We have <u>separated</u> the decision of which risky portfolio to hold from the investor's personal preferences. The preferences come in only when deciding how much of the risk-free asset to hold.

4.4 Tobin's Development of the Separation Theorem

The separation theorem was not an attempt to simplify the investor's solution to Markowitz's Mean-Variance Analysis. Rather, James Tobin's original paper (1958) was intended to provide a more coherent foundation for Keynes's Liquidity Preference Theory of the Interest Rate. It will be argued in this appendix that Keynes's formulation of the liquidity preference theory of interest was extremely weak – both, from the perspective of development of his earlier work in *A Treatise on Money* and the later developments of portfolio theory. Tobin's separation theorem was tangential to Mean-Variance Analysis while directly relating to broader issues within macroeconomic theory and policy.

4.4.1 Keynes's Awkward Money Demand Function

{{{need to write – but the focus will be on:
Difficulty of Monetary Policy to lower the long-term interest rate
Inelastic expectations
Divergence of opinion re 'normal' rate (price)
Portfolio is 'all or nothing' decision}}}

4.4.2 The Separation Theorem in Relation to Liquidity Preference

Tobin considered the case in which the government issued two types of financial assets: money and bonds. Since the government issues both, the risk of default is the same and zero. In order to simplify matters, Tobin assumes that the bond issued by the government is a consol. Why is this assumption a simplification? A consol is a special type of bond – not actually issued by the U.S. government, but has been issued by other governments and can be approximated with a very long term to maturity (e.g., Disney's 100 year bond) – which makes a set yearly payment. The point is that the bond never matures – thus, does not make a final payment (i.e., face-value).

The price and interest rate of a consol are extremely easy to compute. The present value of all the future yearly payments reduces to a nice formula.

$$\text{(A.1)}\quad P = \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \cdots = \frac{C}{r}$$

Where P is the price of the bond, r is the interest rate (or, more specifically, the yield to maturity), and C the yearly coupon payment. Consols are useful to assume when first introducing bonds because it becomes absolutely clear that the price of the bond and interest rate on the bond move inversely. We can solve for the interest rate by cross-multiplying:

(A.2) $r = \frac{C}{P}$

This is similar to a dividend yield (*D/P* where *D* is the yearly dividend payment, in this case an expectation must be formed, and *P* is the price of the share).

The expected rate of return on the bond is composed of the interest rate on the bond (A.2) and the expectation of a capital gain or loss (*g*).

(A.3) $g = \frac{P^e - P}{P} = \frac{P^e}{P} - 1$

The expected rate of return on the bond is therefore written as:

(A.4) $E(r_B) = r + g$

Assuming a martingale probability for the expected price implies that this is equal to the current price.

(A.5) $E(P) = P^e = P$

This implies that the expected rate of return from holding the bond will be equal to the interest rate – hence, the average value of *g* is zero.

(A.6) $E(r_B) = r$

Note, however, that the expected price is being treated as a random variable with mean of zero and a constant standard deviation (e.g., consider it as a random variable with a normal distribution – thus, you need two pieces of information to identify its particular normal curve, the mean and standard deviation). This is important because one tends to forget about the capital gains/loss since it drops out of the expected rate of return calculation – but, it plays an important role in the rest of the analysis.

What about the money asset? We assume for simplicity that money is defined in such a way that it pays zero interest and, of course, has no capital gain/loss. Before moving on, consider the different definitions of money: M1 (currency + checkable deposits), M2 (M1 + small saving accounts), M3 (M2 + large saving accounts). Today, banks do in fact pay interest on checking accounts and have always paid it on saving accounts). Furthermore, Keynes had argued that in some circumstances money should be defined to include short-term government Treasury bills. For Keynes, the essential difference between money and bonds was their price fluctuation. The price of a short-term T-bill will not fluctuate very much with a change in the interest rate. On the other hand, the price of a long-term government bond (such as a consol) would fluctuate greatly with a change in the interest rate. In modern terminology, the percentage change in the price brought about by a one percent change in the interest rate is called the duration. The difference between money and bonds, for Keynes, amounts to the notion that duration is small for money and large for bonds. For us, this amounts to assuming that the capital gains/loss on money is negligible – hence, the variance of this is so low that it can be safely ignored. Thus, money has a zero expected rate of return and variance should be a safe assumption – we can always define the money asset so that it has the characteristics, or very close.

At this point, we can apply the tools developed to handle the Mean-Variance Analysis. The expected rate of return on the portfolio is merely a weighted average of the expected returns for the individual assets.

$$\begin{aligned} & E(r_P) = W_M E(r_M) + W_B E(r_B) \\ & E(r_M) = 0, \\ & E(r_B) = r \rightarrow \\ & E(r_P) = W_B r \end{aligned} \tag{A.7}$$

The variance of the portfolio is not the simple weighted average of the individual asset variances. We have seen that this depends upon the covariance (hence, correlation) between the assets.

$$\sigma_P^2 = (W_M \sigma_M)^2 + (W_B \sigma_B)^2 + 2(W_M \sigma_M)(W_B \sigma_B)\rho$$

This simplifies greatly once we recall that the rate of return on the money asset has zero variance. Hence, we get the following.

$$\text{(A.8)} \quad \sigma_P^2 = (W_B \sigma_B)^2$$

The standard deviation (i.e., risk) of the portfolio is simply the square root of the variance.

$$\text{(A.9)} \quad \sigma_P = W_B \sigma_B$$

All of this has been accomplished with the tools used in constructing the efficient frontier. The difference is that we have introduced an asset without risk and return – what is called a risk-free asset. The introduction of this type of asset carries greater significance than what one might expect. In the present context, our goal is to construct the demand schedule for the risk-free asset.

The available trade-off between risk and return can be developed from equations (A.7) and (A.9). This trade-off will be given by the slope of the line depicting the relationship between risk and return on the portfolio.

$$\text{(A.10)} \quad E(r_P) = \frac{r}{\sigma_B} \sigma_P$$

Notice, this line is derived for solving (A.7) and (A.9) for the proportion of the portfolio held in bonds and equating the result. Graphically, the line is depicted in Figure A.1.

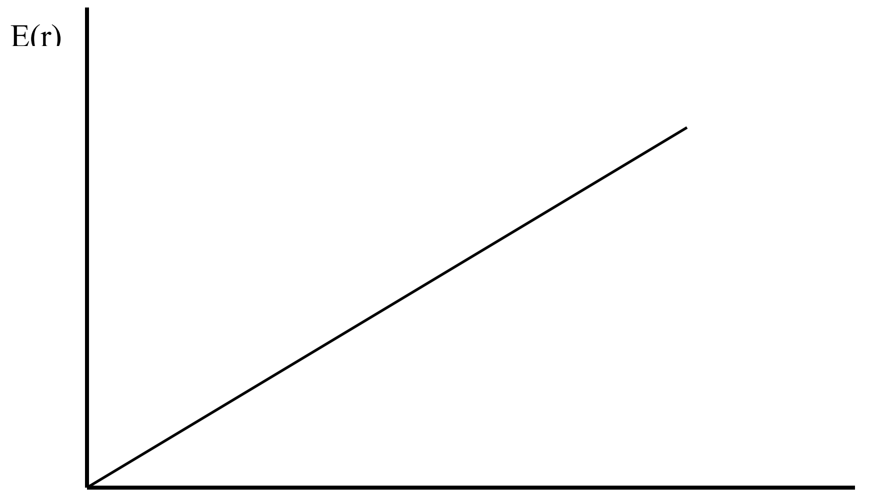


Figure A.1

Notice, the line will tilt upwards for an increase in the interest rate on bonds *or* a decrease in the risk of the bond. The specific place an investor will choose will depend upon how they view the risk and return trade-off. Since it depends upon their subjective preferences for risk and return we can depict their willingness in terms of indifference curves – as done previously.

In order to translate the decision concerning risk and return into the resulting proportion of bonds *and* money held we simply rearrange equation (A.9).

(A.11) $W_B = \dfrac{\sigma_P}{\sigma_B}$

Recall, the proportion of money held in the portfolio will be given by one minus the proportion of bonds held. This relationship is graphed in Figure A.2.

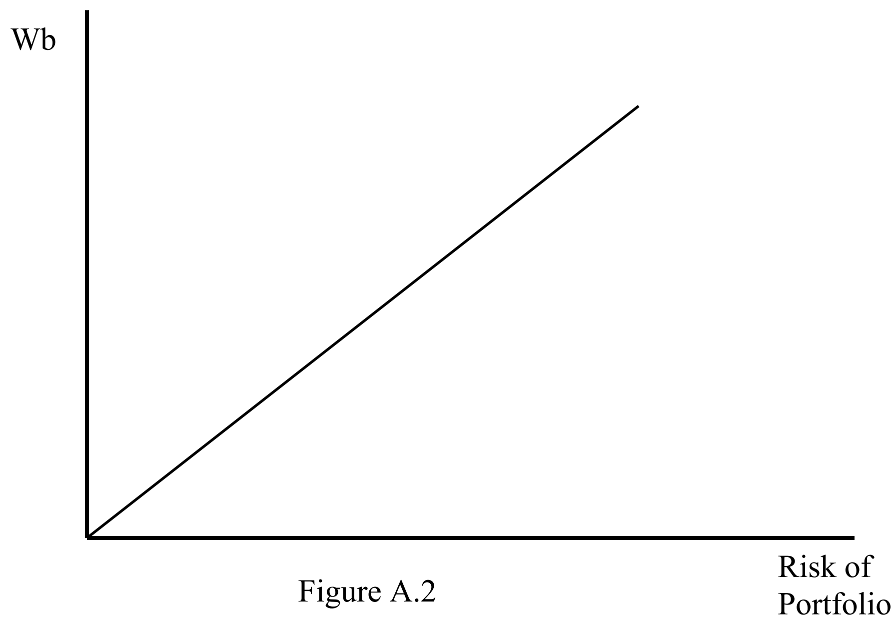


Figure A.2

We can now put everything together to derive a money demand schedule based on portfolio choice (of the Mean-Variance variety). Figure A.3 depicts the result. Notice, the demand for money is read upwards on the lower quadrant. Thus, as the interest rate increases, the top line will tilt upward *normally* leading to an increase in the proportion of bonds held and decrease in the proportion of money – normally is conditional upon how investors for risk, i.e., their indifference curves. We, therefore, derive the demand for money schedule by allowing the interest rate to vary. If the risk of bonds declines, the upper line tilts upward again and the lower line tilts downward!

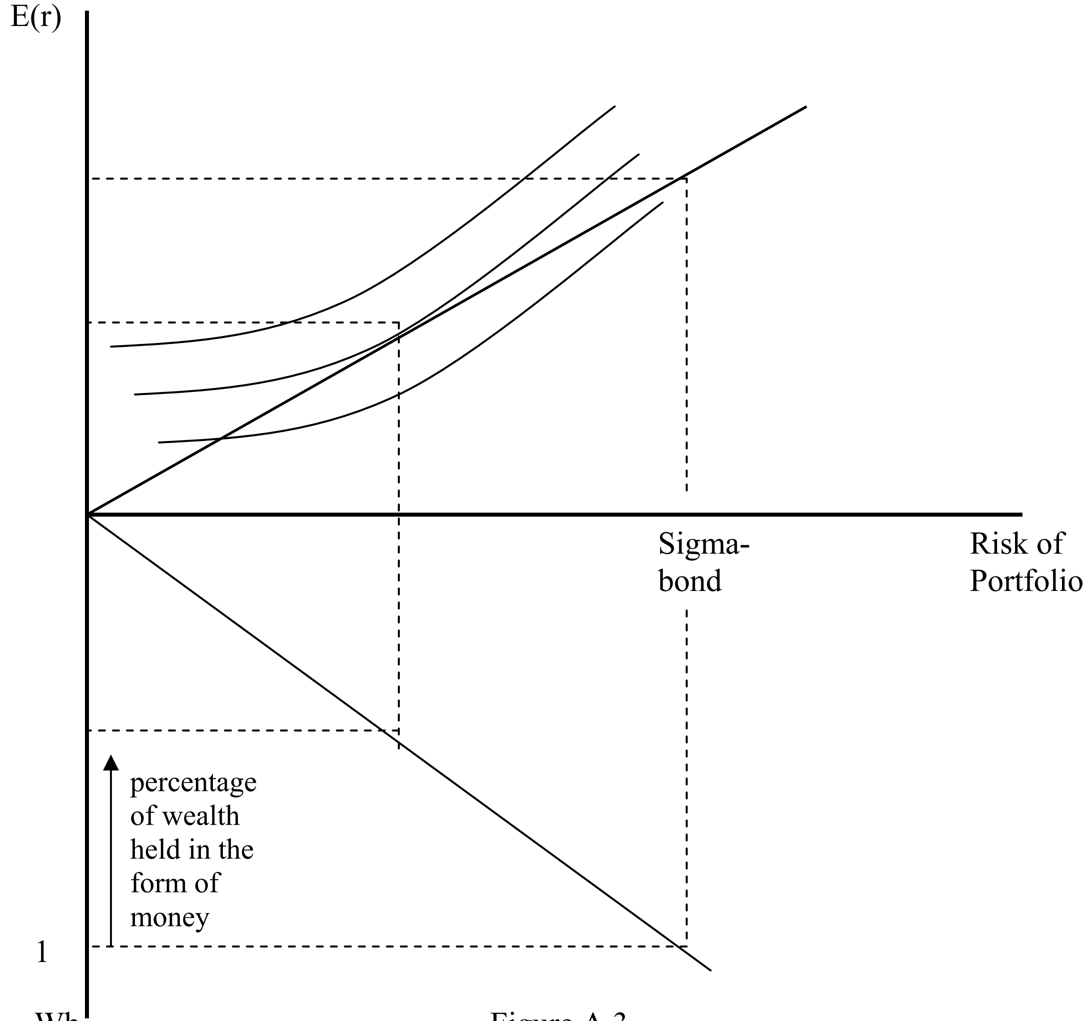

E(r)
Sigma-
bond
Risk of
Portfolio
percentage
of wealth
held in the
form of
money
1
Wb

Figure A.3