Solution Outlines for Chapter 9

# 6: Let \( H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{R}, ad \neq 0 \right\} \). Is \( H \) a normal subgroup of \( GL(2, \mathbb{R}) \)?

No; Show directly by counter example or by multiplying the general case,
\[
\left[ \begin{array}{cc} f & g \\ h & j \end{array} \right] \left[ \begin{array}{cc} a & b \\ 0 & d \end{array} \right] \left( \left[ \begin{array}{cc} f & g \\ h & j \end{array} \right] \right)^{-1},
\]
to see it is not contained in \( H \).

# 8: Viewing \( < 3 > \) and \( < 12 > \) as subgroups of \( \mathbb{Z} \), prove that \( < 3 > / < 12 > \) is isomorphic to \( \mathbb{Z}_4 \). Similarly, prove that \( < 8 > / < 48 > \) is isomorphic to \( \mathbb{Z}_6 \).

Generalize to arbitrary integers \( k \) and \( n \).

First, notice \( < 3 > = \{ \ldots -12, -9, -6, -3, 0, 3, 6, 9, 12, \ldots \} \) and \( 12 > = \{ \ldots -24, -12, 0, 12, 24, \ldots \} \). Now \( < 3 > / < 12 > \) looks like \( \{ -9+ < 12 >, -6+ < 12 >, -3+ < 12 >, < 12 >, 3+ < 12 >, 6+ < 12 >, 9+ < 12 > \} \) since multiples of 12 will be absorbed by \( < 12 > \). Recall \( aH = bH \) if and only if \( b^{-1}a \in H \). Here this tells me that because \(-3)+ -9 = -12, 3+ < 12 > = -9+ < 12 > \). Similarly, \(-3+ < 12 > = 9+ < 12 > \), and \(-6+ < 12 > = 6+ < 12 > \). So, \( < 3 > / < 12 > = \{ < 12 >, 3+ < 12 >, 6+ < 12 >, 9+ < 12 > \} \). Notice that \( 3+ < 12 > \) has order 4 and hence generates all of \( < 3 > / < 12 > \). Thus, \( < 3 > / < 12 > \) is cyclic of order 4, and hence isomorphic to \( \mathbb{Z}_4 \).

Now, consider \( < 8 > / < 48 > \). Similar to before, it is clear that this group consists of \( \{ < 48 >, 8+ < 48 >, 16+ < 48 >, 24+ < 48 >, 32+ < 48 >, 40+ < 48 > \} \). Notice that still similar to before \( 8+ < 48 > \) is a generator of the quotient group and that the group has order 48 divided by 8, or 6. Hence, it is isomorphic to \( \mathbb{Z}_6 \).

In general, suppose \( k \) divides \( n \). Then \( < k > / < n > \) is of the form \( \{ < n >, k+ < n >, 2k+ < n >, \ldots, (n-k)+ < n > \} \). This is clearly cyclic with generator \( k+ < n > \) and has order \( \frac{n}{k} \). Hence \( < k > / < n > \) is isomorphic to \( \mathbb{Z}_{\frac{n}{k}} \).

# 11: Let \( G = \mathbb{Z}_4 \oplus U(4), \) \( H = < (2, 3) >, \) and \( K = < (2, 1) > \). Show that \( G/H \) is not isomorphic to \( G/K \). (This shows that \( H \approx K \) does not imply that \( G/H \approx G/K \).)

For clarity, we write out each of the groups: \( G = \{(0, 1), (1, 1), (2, 1), (3, 1), (0, 3), (1, 3), (2, 3), (3, 3)\}, \)
\( H = \{(2, 3), (0, 1)\}, \) and \( K = \{(2, 1), (0, 1)\}. \) Since \( H \) and \( K \) both have order 2, they are both isomorphic to \( \mathbb{Z}_2 \). Straight forward calculation shows,
\[
G/H = \{ H = (0, 1)H = (2, 3)H, (1, 1)H = (3, 3)H, (2, 1)H = (0, 3)H, (3, 1)H = (1, 3)H \}
\]
and
\[
G/K = \{ K = (0, 1)K = (2, 1)K, (1, 1)K = (3, 1)K, (0, 3)K = (2, 3)K, (3, 3)K = (1, 3)K \}
\]
. Notice that each has 4 elements as expected since \( 4 \times 2 = 8 \).
Consider \((1,3)H: <(1,3)H >= \{(1,3)H, (2,1)H, (3,3)H, (0,1)H\} = G/H\). So, \(G/H\) is cyclic of order 4, and hence is isomorphic to \(Z_4\).

However, observe that \(G/K\) is not cyclic since \(<(0,1)K >= \{K\}, <(1,1)K >= \{(1,1)K, (2,1)K\}, <(0,3)K >= \{(0,3)K, (0,1)K\}\) and \(<(3,3)K >= \{(3,3)K, (2,1)K\}\). In fact, we recognize that this structure is the Klein-4 group, \(Z_2 \oplus Z_2\). Hence \(G/H \neq G/K\).

**# 13: Prove that a factor group of an Abelian group is Abelian.**

Let \(G\) be an Abelian group and consider its factor group \(G/H\), where \(H\) is normal in \(G\). Let \(aH\) and \(bH\) be arbitrary elements of the quotient group. Then \(aHbH = (ab)H = (ba)H = bHaH\) because \(G\) is Abelian. Hence the factor group is also Abelian.

**# 14: What is the order of the element \(14+ < 8 >\) in the factor group \(Z_{24}/ < 8 >\)?**

For completeness, observe \(<8> = \{8, 16, 0\}\) and \(Z_{24}/ < 8 > = \{<8>, 1+ <8>, 2+ <8>, 3+ <8>, 4+ <8>, 5+ <8>, 6+ <8>, 7+ <8>\}. Now let’s observe \(14+ <8>: \)
\[14+ <8> + (14+ <8>) = 28+ <8> = 4+ 8, (14+ <8>) + (4+ <8>) = 18+ <8> = 2+ <8>, (14+ <8>) + (2+ <8>) = 16+ <8> = <8>\]

Hence the order of \(14 + 8\) is 4.

**# 16: Recall that \(Z(D_6) = \{e, r^3\}\). What is the order of the element \(rZ(D_6)\) in the factor group \(D_6/Z(D_6)\)?**

Notice that problem 16 here is rewritten in terms of generators and relations. Now it is clear that the order of \(rZ(D_6)\) is 3 since \(r^3 \in Z(D_6)\).

**# 17: Let \(G = Z/ <20>\) and \(H = <4> / <20>\). List the elements of \(H\) and \(G/H\).**

Observe: \(<4> = \{\ldots, -8, -4, 0, 4, 8, 12, \ldots\}\) and \(<20> = \{\ldots, -40, -20, 0, 20, 40, 60, \ldots\}\). Hence \(H = \{<20>, 4+ <20>, 8+ <20>, 12+ <20>, 16+ <20>\} \approx Z_5\).

Now notice that \(G = \{<20>, 1+ <20>, 2+ <20>, \ldots, 19+ <20>\} \approx Z_{20}\). So \(G/H = \{0+ <20> + H, 1+ <20> + H, 2+ <20> + H, 3+ <20> + H\} \approx Z_4\).

**# 19: What is the order of the factor group \((Z_{10} \oplus U(10))/ <(2,9)>)?**

The order of the factor group is \(\left|\frac{Z_{10} \oplus U(10)}{<(2,9)>}\right| = \frac{10 \times 4}{\text{lcm}(2,|9|)} = \frac{40}{18} = 4\).

**# 21: Prove that an Abelian group of order 33 is cyclic.**

Let \(G\) be an Abelian group of order 33. By Theorem 9.5, there exists an element of \(G\), say \(a\), such that \(|a| = 3\) and an element of \(G\), say \(b\), such that \(|b| = 11\). Since \(G\) is Abelian, \((ab)^{33} = a^{33}b^{33} = e\) so the order of \(ab\) divides 33. However, it is clear \(|ab|\) is not 1, 3, or 11. Hence \(|ab| = 33\) so \(ab \in G\) generates \(G\), and \(G\) is cyclic.
# 23: Determine the order of \((\mathbb{Z} \oplus \mathbb{Z})/ < (4, 2) >\). Is the group cyclic?

Notice that \((1, 1)+ < (4, 2) >\) has infinite order [Why? Suppose it is of finite order, say \(n\). Then \((n, n) \in < (4, 2) >\) which means \((n, n) = k(4, 2)\) for some \(k\). So \(k = n/4 = n/2\) or \(4n = 2n\) which means \(n = 2n\) so \(n = 0\) since \(n\) is an integer.]. Hence the group \((\mathbb{Z} \oplus \mathbb{Z})/ < (4, 2) >\) also has infinite order.

If the quotient group is cyclic, it must be isomorphic to \(\mathbb{Z}\) (from previous work) so every non-identity element should have infinite order. However, \((6, 3)+ < (4, 2) >\) has order 2. Hence, it is not cyclic.

# 24: The group \((\mathbb{Z}_4 \oplus \mathbb{Z}_{12})/ < (2, 2) >\) is isomorphic to one of \(\mathbb{Z}_8\), \(\mathbb{Z}_4 \oplus \mathbb{Z}_2\), or \(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2\). Determine which one by elimination.

Observe that \(H = < (2, 2) > = \{(2, 2), (0, 4), (2, 6), (0, 8), (2, 10), (0, 0)\}\) (which has order 6 as expected). Let \(G = (\mathbb{Z}_4 \oplus \mathbb{Z}_{12})/ < (2, 2) >\). Then \(G = \{H, (1, 0)H, (0, 1)H, (1, 1)H, (0, 2)H, (0, 3)H, (3, 0)H, (1, 3)H\}\) and these cosets have orders 1, 2, 4, 4, 2, 4, 4, and 2 respectively. Hence, \(G\) is not cyclic and not isomorphic to \(\mathbb{Z}_8\). Further, since there is an element of order 4, \(G\) is not isomorphic to \(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2\). Hence, \(G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2\).

# 25: Let \(G = U(32)\) and \(H = \{1, 31\}\). The group \(G/H\) is isomorphic to one of \(\mathbb{Z}_8\), \(\mathbb{Z}_4 \oplus \mathbb{Z}_2\), or \(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2\). Determine which one by elimination.

First, we know that the order of \(U(32) = 2^5 - 2^4 = 16\) so \(G/H\) has order \(16/2 = 8\) as anticipated.

Consider \(3H = \{3, 29\} \in G/H: < 3H > = \{3H, 9H, 27H, 17H, 19H, 25H, 11H, H\}\) so the order of \(3H\) is 8. Hence \(G/H = < 3H > \approx \mathbb{Z}_8\).

# 27: Let \(G = U(16), H = \{1, 15\}\) and \(K = \{1, 9\}\). Are \(H\) and \(K\) isomorphic? Are \(G/H\) and \(G/K\) isomorphic?

It is obvious that \(H \approx K \approx \mathbb{Z}_2\). Now, we need to check if \(G/H\) and \(G/K\) are isomorphic. We know that each has order 4 and that there are only two such groups. Consider \(3H: < 3H > = \{3H, 9H, 11H, H\}\) so 3 \(H\) generates \(G/H\) and \(G/H \cong \mathbb{Z}_4\). Now observe \(G/K: < K > = \{K\}, < 3K > = \{3K, K\}, < 5K > = \{5K, K\}\) and \(< 7K > = \{7K, K\}\). Thus \(G/K \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2\) and \(G/K \not\cong G/H\).

# 37: Let \(G\) be a finite group and let \(H\) be a normal subgroup of \(G\). Prove that the order of the element \(gH\) in \(G/H\) must divide the order of \(g\) in \(G\).

Let \(|g| = n\). Then \((gH)^n = g^nH = eH = H\) so \(|gH|\) must divide \(n\).

# 38: Let \(H\) be a normal subgroup of \(G\) and let \(a\) belong to \(G\). If the element \(aH\) has order 3 in the group \(G/H\) and \(|H| = 10\), what are the possibilities for the order of \(a\)?
First, $|G| = |aH| 	imes |H| = 3 \times 10 = 30$. So $|a|$ divides 30. But we also know, by the previous problem, that 3 also has to divide $|a|$. Hence the possible orders for $a$ are 3, 6, 15, and 30.

40: Let $\phi$ be an isomorphism from a group $G$ onto a group $\bar{G}$. Prove that if $H$ is a normal subgroup of $G$, then $\bar{\phi}(H)$ is a normal subgroup of $\bar{G}$.

Let $H$ be normal in $G$. We want to show $y\phi(H)y^{-1} \subseteq \phi(H)$ for all $y \in \bar{G} = \phi(G)$. Since $y \in \phi(G)$, there exists an $x \in G$ such that $y = \phi(x)$. Thus $y\phi(H)y^{-1} = \phi(x)\phi(H)(\phi(x))^{-1} = \phi(xHx^{-1}) = \phi(H)$ since $H$ is normal in $G$, and we are done.

42: An element is called a square if it can be expressed in the form $b^2$ for some $b$. Suppose that $G$ is an Abelian group and $H$ is a subgroup of $G$. If every element of $H$ is a square and every element of $G/H$ is a square, prove that every element of $G$ is a square. Does your proof remain valid when “square” is replaced by “$n$th power” where $n$ is any integer?

Let $G$ be an Abelian group, $H$ be a subgroup of $G$ and every element of both $H$ and $G/H$ be a square. Suppose $g \in G$. Since $g \in G$, $gH \in G/H$. But all elements of $G/H$ are squares so there exists an $aH \in G/H$ such that $gH = (aH)^2 = a^2H$. By properties of cosets, we now have that $(a^2)^{-1}g \in H$. But every element in $H$ is a square so there exists a $b \in H$ such that $(a^2)^{-1}g = b^2$. Solving for $g$ we see $g = a^2b^2 = (ab)^2$ since $G$ is Abelian. But this means that $g$ is a square. Hence every element of $G$ is a square.

Notice that this did not depend on a property of 2 so the proof remains valid when 2 is replaced by $n \in \mathbb{Z}$.

46: Show that $D_{13}$ is isomorphic to $Inn(D_{13})$.

First, recall that $Z(D_{13}) = \{e\}$. Now, we know that $Inn(D_{13}) \approx D_{13}/Z(D_{13}) = D_{13}$.

49: Suppose that $G$ is a non-Abelian group of order $p^3$ where $p$ is prime and $Z(G) \neq \{e\}$. Prove that $|Z(G)| = p$.

First recall that $Z(G)$ is normal in $G$. Since $G$ is non-Abelian, $Z(G)$ does not have order $p^3$. Farther, since $Z(G)$ is a non-trivial subgroup, it’s order is not 1 and divides $p^3$ so it has order $p$, or $p^2$.

Suppose that the order of $Z(G)$ is $p^2$. Then $|G/Z(G)| = p$ and hence the quotient group $G/Z(G)$ is cyclic. But this implies, by Theorem 9.3, that $G$ is Abelian, which is a contradiction. Hence $|Z(G)| = p^2$.

50: If $|G| = pq$ where $p$ and $q$ are primes that are not necessarily distinct, prove that $|Z(G)| = 1$ or $pq$.

Let $|G| = pq$, as above. Since $Z(G)$ is a normal subgroup of $G$, $|Z(G)| = 1, p, q$, or $pq$. If $G$ is Abelian, $|Z(G)| = pq$. 
Assume $G$ is not Abelian. Without loss of generality, let $|Z(G)| = p$. Then $|G/Z(G)| = q$, which is prime. Hence $|G/Z(G)|$ is cyclic and $G$ is Abelian. But this is a contradiction. Hence $|Z(G)| = 1$.

# 51: Let $N$ be a normal subgroup of $G$ and let $H$ be a subgroup of $G$. If $N$ is a subgroup of $H$, prove that $H/N$ is a normal subgroup of $G/N$ if and only if $H$ is a normal subgroup of $G$.

Let $N$ be a normal subgroup of $G$ and let $H$ be any subgroup of $G$. Assume $N \subseteq H$.

“⇒” Let $H/N$ be normal in $G/N$. Then for all $gN \in G/N$ and $hN \in H/N$, $(gN)(hN)(gN)^{-1} = (ghg^{-1})N \in H/N$. Thus $ghg^{-1}N = h'N$ for some $h'in H$. Hence $ghg^{-1} = h'n$ for some $n \in N$. But $h' \in H$ and $n \in H$ so $h'n \in H$. Hence $gHg^{-1} \subseteq N$. Thus $H$ is normal in $G$.

“⇐” The argument above reverses.

# 56: Show that the intersection of two normal subgroups of $G$ is a normal subgroup of $G$. Generalize.

Let $H$ and $K$ be normal subgroups of $G$. Let $x \in H \cap K$ and $g \in G$. Since $x \in H$, $gxg^{-1}$ is in $H$. Similarly, $gxg^{-1}$ is in $K$. Thus $gxg^{-1}$ is in $H \cap K$ for all $g \in G$ and $x \in H \cap K$. Thus, $H \cap K$ is normal in $G$. Note that in a previous chapter we showed that $H \cap K$ is a subgroup of $G$, which completes the proof.

# 61: Let $H$ be a normal subgroup of a finite group $G$ and let $x \in G$. If $\gcd(|x|, |G/H|) = 1$, show that $x \in H$.

Let $\gcd(|x|, |G/H|) = 1$ as above. From an earlier problem we know that $|xH|$ must divide $|x|$, so $\gcd(|xH|, |G/H|)$ must also be $1$. But we also know that $|xH|$ must divide $|G/H|$ because $xH$ is an element of this group. Hence $|xH| = 1$ so $xH = H$, which implies $x \in H$.

# 63: If $N$ is a normal subgroup of $G$ and $|G/N| = m$, show that $x^m \in N$ for all $x$ in $G$.

Let $x \in G$ and $|G/N| = m$. Then $x^mN = (xN)^m = (xN)^{|G/N|} = N$ so $x^m \in N$.

# 68: Recall that a subgroup $N$ of a group $G$ is called characteristic if $\phi(N) = N$ for all automorphisms $\phi$ of $G$. If $N$ is a characteristic subgroup of $G$, show that $N$ is a normal subgroup of $G$.

Let $N$ be a characteristic subgroup of $G$. Then $\phi(N) = N$ for all automorphisms of $G$. In particular, $\phi_g(N) = N$ when $\phi_g$ is the conjugation map by $g$. Thus $gNg^{-1} = N$ for all $g \in G$. So $N$ is normal in $G$. 
Team Problem Solutions for Ch 9

# 10: Let $H = \{(1), (12)(34)\}$ in $A_4$.

a. Show that $H$ is not normal in $A_4$.

We know that $(123)H = \{(123), (134)\}$ and $H(123) = \{(123), (324)\}$. These are not equal so $H$ is not normal in $A_4$.

b. Referring to the multiplication table for $A_4$ in Table 5.1 on page 111, show that, although $\alpha_6H = \alpha_7H$ and $\alpha_9H = \alpha_{11}H$, it is not true that $\alpha_6\alpha_9H = \alpha_7\alpha_{11}H$. Explain why this proves that the left cosets of $H$ do not form a group under coset multiplication.

$\alpha_6\alpha_9H = (243)(132)H = (12)(34)H = H$ and $\alpha_7\alpha_{11}H = (142)(234)H = (14)(23)H \neq H$.

This shows that multiplication is not well defined for these cosets and hence the left cosets of $H$ do not form a group under coset multiplication. This does not surprise us since we know that normality was required for well-defined.

# 47: Suppose that $N$ is a normal subgroup of a finite group $G$ and $H$ is a subgroup of $G$. If $|G/N|$ is prime, prove that $H$ is contained in $N$ or that $NH = G$.

Let $N$ be a normal subgroup of a finite group $G$, and $H$ be any subgroup of $G$. Let $|G/N| = p$, a prime. Now we know that $N \subseteq NH \subseteq G$. Therefore, $p = |G : N| = |G : NH| \times |NH : N|$. Thus $|G : NH|$ is $p$ or $1$. If $|G : NH| = 1$, then $G = NH$. If $|G : NH| = p$, then $|NH : N| = 1$ so $NH = N$, which means that $H \subseteq N$.

# 65: If $G$ is non-Abelian, show that $\text{Aut}(G)$ is not cyclic.

Proof. Suppose not. Let $\text{Aut}(G)$ be cyclic. Then $\text{Inn}(G)$ is cyclic since $\text{Inn}(G)$ is a subgroup of $\text{Aut}(G)$ and subgroups of cyclic groups are cyclic. We know that $\text{Inn}(G) \approx G/Z(G)$ so $G/Z(G)$ must be cyclic. But this implies that $G$ is Abelian, which is a contradiction. Thus $\text{Aut}(G)$ is not cyclic. $\square$